THE UNIVERSITY OF ALBERTA

# A FINITE STRAIN THEORY FOR
# ELASTIC-PLASTIC DEFORMATION

BY

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ABSTRACT

A brief review of the classical theories of rigid perfectly plastic and elastic-perfectly plastic continua is presented and the invariance requirements of constitutive equations, which lead to the kinematically correct form of the Prandtl-Reuss equations are discussed.

The kinematics and thermodynamics of elastic-plastic deformation of materials which may undergo finite elastic strains are discussed and a plastic flow rule obeying the necessary invariance requirements is proposed for these materials. Relevant topics in the theory of finite strain elasticity are reviewed.

Two intepretations of the von Mises yield condition are given for a material which may undergo finite elastic strains, and the associated plastic flow rules are found.

Numerical solutions are obtained for the stress deviator tensor in a cuboid undergoing simple shear. The two elastic-perfectly plastic materials considered exhibit neo-Hookean elastic behavior prior to yielding and during unloading. Both interpretations of the von Mises yield condition and the proposed plastic flow rule are used.

The numerical results found for the simple shear problem are used to obtain the stresses in twisted circular cylinders composed of the same materials.

A solution is also found for the stresses in a thick spherical

shell expanded by an internal pressure and composed of a neo-Hookean elastic-perfectly plastic material obeying any isotropic yield condition. The residual stresses resulting from unloading from an elastic-plastic state are also found.

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# CHAPTER I

# PERFECTLY PLASTIC AND ELASTIC-PERFECTLY PLASTIC CONTINUA

## 1.1 Introduction

A branch of applied mathematics called continuum mechanics has been developed which attempts to describe the macroscopic behavior of deforming materials by considering idealized mathematical models. It is not surprising in view of the natural division of materials into solids and fluids, that the two most well established branches of this field are classical elasticity and hydrodynamics. The need however, for more elaborate theories, valid for the wide range of applications of materials, and to describe the numerous new materials which have been developed has accelerated the growth of continuum mechanics and has led to several new branches.

One such development is the mathematical theory of plasticity. The plastic deformation of metals is the basis of all metal forming processes and consequently it is of considerable technological interest.

It is observed in a simple tension test of a material such as steel that elastic behavior occurs under increasing load only up to a certain point called the yield point after which plastic flow occurs with continuing deformation resulting with little increase in load. In contrast to the viscous flow of fluids, the resistance to shear deformation of plastically flowing materials appears to be almost inde-

pendent of the rate at which the deformation occurs.

Early work in plasticity, such as that of Coulomb [1]* in 1773, dealt mainly with the failure of soils. In 1864 Tresca [2] published the results of some experiments on the extrusion of metals and concluded from these experiments that the yield point is reached when the maximum shearing stress reaches a critical value. Soon after, St. Venant [3] suggested that for plane plastic strain the principal axes of stress and plastic strain increment coincide. Lèvy [4] in 1870 extended this idea to three dimensions and proposed a coaxial relation between the stress deviator tensor and the strain increment tensor. The same relation was suggested independently by von Mises [5] in 1913. In 1928 von Mises [6] extended his theory to consider a perfectly plastic solid with an arbitrary regular yield function $f(\sigma_{ij})$.

Hencky [7] proposed a so called total deformation theory in 1924 in which the stress and the total plastic strain are related rather than the stress and the plastic deformation rate or strain increment as in the incremental or flow theory of St. Venant and Lèvy. Valid objections have been raised against total deformation theories by Handelman, Lin, and Prager [8] and by Hill [9]. These theories have often been used however, especially in Russia, and under some conditions they give satisfactory results.

The incorporation of elastic strains in the incremental or flow theory was first done by Prandtl [10]in 1924 for plane problems and

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*The numbers in square brackets refer to references listed in the Bibliography at the end of this thesis.

by Reuss [11] in 1930 for three dimensional problems.

In the period following World War II until the present, much work has been done to put the theory of plasticity on a sound mathematical and physical foundation. Several reference books have been written [12], [13], [14], [15], [16], [17], and a number of extensive review papers have appeared [18], [19].

Some recent developments in the theory of elastic-plastic continua have involved the consideration of finite deformations. For example the kinematically correct form of the Prandtl-Reuss equations, valid for large total deformations with small elastic strains was first presented by Thomas [15], [20] in 1955.

Classical elastic-plastic theory, including the Prandtl-Reuss equations, is not applicable to the consideration of elastic-plastic flow in the presence of finite elastic strain. One example of this is the elastic-plastic flow of a metal which is subjected to an hydrostatic pressure which is sufficient to produce finite elastic volume change. Such pressures are produced in certain explosive forming processes.

The results of some preliminary experiments at the Department of Mechanical Engineering, University of Alberta, indicate that several types of a urethane elastomer (duPont Adiprene) exhibit permanent set after finite elastic shear strains. Cubes of this material, of 5/8 inch side, were compressed slowly between lubricated parallel flat dies. Tests conducted in the temperature range 70 - 80°F showed significant rate effects with the recovery of a major portion of the residual

deformation occurring over a period of several days. At temperatures around 32°F however the residual deformation which occurred after reductions of approximately 30% appeared to be permanent. The elastic behavior of these materials is approximately neo-Hookean and volume measurements of deformed specimens showed that they are very nearly incompressible.

These experiments were not extensive and are not part of the work of this thesis. They are mentioned however because they indicate a practical and important area for more extensive experimental work which may or may not justify some of the assumptions made in this thesis.

High energy rate metal forming problems and the possibility that an elastic-plastic theory may, although not yet experimentally verified, provide an approximation to the behavior of some elastomers at certain temperatures has motivated the development of a number of theories for elastic-plastic continua which may undergo finite elastic strains.

One of the first works which may be applicable to such problems appears to have been by Sedov [21]. A theory for rate independent materials in which the stress is a functional of the entire deformation history has been presented by Pipkin and Rivlin [22]. Isothermal deformations only are considered. A theory which is not restricted in this manner has been proposed by Green and Naghdi [23], [24]. It is modelled on the classical flow theory and makes considerable use of thermodynamic considerations. Lee [25], [26], [27] has also suggested a flow theory for elastic-plastic materials under finite elastic and plastic strains which differs from that of Green and Naghdi. The most recent work is

that of Freund [28]. It is an extension of the work of Lee and introduces a plastic rate of deformation tensor which differs from that suggested by Lee and that developed in Chapter II.

In this thesis the kinematics for a general theory for materials which may undergo finite elastic and plastic strains is discussed in Chapter II. Following the discussion in section 3.7 the development of the theory is restricted to materials which are elastically isotropic. An argument based on the assumption that the materials considered are elastically incompressible and obey Drucker's postulate [30] leads to a general plastic flow rule from which two specific flow rules are found, one of which is associated specifically with materials whose elastic behavior is neo-Hookean. A simplification which results if plastic isotropy is assumed is indicated in section 4.4. Three boundary value problems are solved. These are the isothermal simple shear of a cuboid, torsion of a circular cylinder, and expansion of a spherical shell, the respective deformations being homogeneous and rotational, inhomogeneous and rotational, and inhomogeneous and irrotational. The materials considered in the problems are isotropic, incompressible, and exhibit neo-Hookean elastic behavior.

## 1.2 Constitutive Equations for an Hookean Elastic Material

Let $u_i$ be the components of the displacement vector of a particle of an Hookean elastic material, referred to a fixed Cartesian coordinate system. The small strain tensor referred to this coordinate system has components

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{1.2.1}$$

The constitutive equation for an Hookean elastic material is

$$\sigma_{ij} = \frac{E}{1+\nu} e_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)} e_{kk} \delta_{ij} \tag{1.2.2}$$

where $\sigma_{ij}$ are the components of the Cauchy stress tensor, E is Young's modulus, and $\nu$ is Poisson's ratio [29].

Substituting the components of the stress deviator

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$$

and the strain deviator

$$e'_{ij} = e_{ij} - \frac{1}{3} e_{kk} \delta_{ij}$$

in equation (1.2.2) gives

$$e'_{ij} = \frac{1}{2\mu} \sigma'_{ij} , \ e_{kk} = - p/K \tag{1.2.3a,b}$$

where $\mu = \frac{E}{2(1+\nu)}$ is the modulus of rigidity, $K = \frac{E}{3(1-2\nu)}$ is the bulk modulus and $p = \frac{-1}{3} \sigma_{kk}$. For an incompressible material $e'_{ij} = e_{ij}$ and only the first of equations (1.2.3) is required. The hydrostatic part of the stress is then determined from the equations of motion.

## 1.3 Constitutive Equations for a Rigid Perfectly Plastic Material

The rigid perfectly plastic material is a hypothetical material which remains rigid under loading until the stresses satisfy a yield

condition. Then irreversible, rate independent deformation occurs at constant stress. The model is a reasonable approximation to some structural materials such as steel which have a well defined yield point and for which the modulus of rigidity is large compared to the yield stress. It is not suitable however for problems in which the plastic strains are constrained to be of the same order of magnitude as the elastic strains.

Fundamental to the theory is the concept of a yield condition. It is postulated that there exists a function of the stress and temperature, $f(\sigma_{ij},T)$, called the yield function which has the properties

$$f \leq 0 ,$$

$$d_{ij}^{(p)} = 0 \text{ if } f < 0 \text{ or } f = 0 \text{ and } df < 0 ,$$

and

$$|d_{ij}^{(p)}| \geq 0 \text{ if } f = 0 \text{ and } df = 0 ,$$

where $d_{ij}^{(p)}$ are the components of the plastic rate of deformation tensor given by

$$d_{ij}^{(p)} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) ,$$

$v_i$ being the components of velocity at a point.

In nine dimensional stress space the yield condition $f(\sigma_{ij},T) = 0$ defines a closed surface. Deformation is possible only under stress systems

which lie on the surface, and stress points outside the surface are not admissible.

Consider a body in equilibrium under an homogeneous stress system $\sigma_{ij}^{o}$ which is inside or on the yield surface. Suppose that an independent external agency applies and removes isothermally an additional stress to the body. Drucker's postulate [30] states that the work done by the external agency during the complete cycle is non-negative for a perfectly plastic material. That is, the external agency cannot extract useful work from the system by applying and removing additional stresses. Drucker's postulate defines a class of material models which have been found to provide an acceptable approximation for many structural materials.

Two important results follow from this postulate. Let the stress point $\sigma_{ij}^{o}$ lie inside the yield surface. With the addition and removal of the external agency the stress point follows a closed path in stress space. Suppose that during the time interval $[t, t + \delta t]$ the stress path lies on the yield surface. That is, during this time interval, the body has yielded. The work per unit volume done by the total stress system during the complete cycle is

$$W = \int_{t}^{t+\delta t} \sigma_{ij}\, d_{ij}^{(p)}\, dt \ ,$$

since for a rigid plastic material deformation can occur only when the stress point lies on the yield surface and in general for an elastic-

plastic material the net work done on the elastic strains in a closed cycle is zero.

Similarly the work done per unit volume by the original stress $\sigma^o_{ij}$ during the complete cycle is

$$W^o = \int_t^{t+\delta t} \sigma^o_{ij} d^{(P)}_{ij} dt .$$

Drucker's postulate states that

$$W - W^o = \int_t^{t+\delta t} (\sigma_{ij} - \sigma^o_{ij}) d^{(P)}_{ij} dt \geq 0 .$$

It follows that

$$(\sigma_{ij} - \sigma^o_{ij}) d^{(P)}_{ij} \geq 0 , \qquad (1.3.1)$$

where $\sigma^o_{ij}$ is any stress point lying on or inside the yield surface and $\sigma_{ij}$ is any stress point on the yield surface with an associated plastic rate of deformation tensor $d^{(P)}_{ij}$ .

Equation (1.3.1) which is known as the maximum work inequality implies that the yield surface is convex and that a vector in stress space with components $d^{(P)}_{ij}$ , multiplied by a suitable scalar to give the units of stress, is normal to the yield surface at the point $\sigma_{ij}$ or lies between adjacent normals if $\sigma_{ij}$ is at a corner of an irregular yield surface.

Thus for a regular yield function

$$d_{ij}^{(p)} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \tag{1.3.2}$$

where $\lambda$ is a non-negative scalar factor of proportionality and $\lambda = 0$ if $f < 0$ or $\dot{f} < 0$ . The flow rule (1.3.2) implies that if yielding is not influenced by the hydrostatic part of the stress tensor the plastic deformation is isochoric, that is $d_{kk}^{(p)} = 0$.

For an isotropic rigid plastic material the form of the yield function at a point must be independent of the orientation of the material element considered so that the yield function depends only on the invariants of the stress tensor. Furthermore the experimentally observed fact that yielding is almost independent of moderate hydrostatic pressure [31] results in the usual assumption that for an isotropic solid f is a function of the stress deviator invariants

$$J_2' = \frac{1}{2} \sigma_{ij}' \sigma_{ij}' ,$$

and

$$J_3' = \frac{1}{3} \sigma_{ij}' \sigma_{jk}' \sigma_{ki}' .$$

In general, the flow rule associated with the yield function f for an incompressible isotropic perfectly plastic material is

$$d_{ij}^{(p)} = \lambda[\sigma_{ij}' \frac{\partial f}{\partial J_2'} + (\sigma_{ik}'\sigma_{kj}' - \frac{2}{3} J_2' \delta_{ij}) \frac{\partial f}{\partial J_3'}] . \tag{1.3.3}$$

For the von Mises yield condition

$$\frac{1}{2}\sigma'_{ij}\sigma'_{ij} - k^2 = 0 \ , \qquad (1.3.4)$$

where k is the yield stress in pure shear, the flow rule (1.3.3) becomes

$$d_{ij}^{(P)} = \lambda\sigma'_{ij} \ ,$$

from which

$$d_{ij}^{(P)}d_{ij}^{(P)} = \lambda^2\sigma'_{ij}\sigma'_{ij} \ . \qquad (1.3.5)$$

From equations (1.3.4) and (1.3.5) the following expression for $\lambda$ is obtained,

$$\lambda = \frac{1}{k}\sqrt{\frac{d_{ij}^{(P)}d_{ij}^{(P)}}{2}}$$

and thus

$$d_{rs}^{(P)} = \frac{1}{k}\sqrt{\frac{d_{ij}^{(P)}d_{ij}^{(P)}}{2}}\,\sigma'_{rs} \ .$$

As mentioned by Thomas [32], there is no one-one relationship between the stress deviator and the plastic rate of deformation tensors unlike the constitutive equations for a viscous fluid.

## 1.4 Prandtl-Reuss Equations for Small Total Deformation

The classical elastic-plastic material is one for which the total deformation rate may be written as the sum of a plastic and an elastic part which are obtained respectively from the rigid plastic flow rule and the material derivative of the Hookean elastic stress strain law.

Suppose that during a time increment dt the increment of stress at a point in an elastic-plastic material is $d\sigma_{ij}$ referred to a fixed Cartesian coordinate system $ox_i$ and denote by $du_i$ the increment of displacement which occurs during this time increment.*

The total strain increment then is

$$de_{ij} = \frac{1}{2}\left(\frac{\partial(du_i)}{\partial x_j} + \frac{\partial(du_j)}{\partial x_i}\right) . \qquad (1.4.1)$$

It is assumed that $de_{ij}$ may be written as the sum of an elastic and a plastic strain increment** so that

$$de_{ij} = de_{ij}^{(e)} + de_{ij}^{(p)}$$

and thus

$$d_{ij} = d_{ij}^{(e)} + d_{ij}^{(p)}$$

where $d_{ij}^{(e)}$ and $d_{ij}^{(p)}$ are the elastic and plastic rate of deformation

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* The components $du_i$ are given by $du_i = v_i dt$ where $v_i$ is the velocity at the point considered.

**Although $de_{ij}$ must be derivable from an incremental displacement field this is not necessarily true for $de_{ij}^{(e)}$ and $de_{ij}^{(p)}$ .

tensors respectively.

An expression for the elastic rate of deformation tensor is obtained by taking the material derivative of equations (1.2.3) to give

$$d_{ij}^{(e)} = \frac{1}{2\mu}\, \dot{\sigma}'_{ij} - \frac{1}{3K}\, \dot{p}\, \delta_{ij} \ .$$

The plastic rate of deformation tensor is given by equation (1.3.2) and thus the total deformation rate tensor during loading is

$$d_{ij} = \frac{1}{2\mu}\, \dot{\sigma}'_{ij} - \frac{1}{3K}\, \dot{p}\, \delta_{ij} + \lambda\, \frac{\partial f}{\partial \sigma'_{ij}} \ .$$

In particular for an elastic-perfectly plastic material obeying the von Mises yield condition (1.3.4) the constitutive equation is

$$d_{ij} = \frac{1}{2\mu}\, \dot{\sigma}'_{ij} - \frac{1}{3K}\, \dot{p}\, \delta_{ij} + \lambda\sigma'_{ij} \ . \qquad (1.4.2)$$

An expression for $\lambda$ may be obtained by multiplying equation (1.4.2) by $\sigma'_{jk}$ and contracting. This gives

$$\lambda = \frac{d_{ij}\sigma'_{ij}}{2k^2}$$

with $\lambda = 0$ if $\sigma'_{ij}\, \sigma'_{ij} < 2k^2$ or if $d_{ij}\, \sigma'_{ij} < 0$ .

Equations (1.4.2) are called the Prandtl-Reuss equations and for an incompressible material they become

$$d_{ij} = \frac{1}{2\mu} \dot{\sigma}'_{ij} + \lambda \sigma'_{ij} . \tag{1.4.3}$$

## 1.5 Invariance Requirements for Constitutive Equations

The constitutive equations of a material are essentially a mathematical description of the deformation properties of the material and must satisfy certain invariance requirements to be physically meaningful.

The first requirement is called coordinate invariance. The deformation characteristics of a material must be independent of the coordinate system used to describe the system. This requirement is satisfied by representing correctly all physical quantities by their components as scalars, vectors, and tensors referred to some coordinate system.

The second invariance principle may be stated in two slightly different but equivalent forms called the principle of material indifference or material objectivity and the principle of isotropy of space [33], [34], [35].

Consider a body whose motion is given by $x_i(X_K,t)$ referred to a coordinate system in the reference frame S. The quantities $x_i$ are the coordinates at time t of a material particle which occupied the point with coordinates $X_K$ at some reference time $t_o$. Consider the following equations,

$$\tilde{x}_i(X_K,t) = C_i(t) + Q_{ij}(t)\, x_j(X_K,t) , \tag{1.5.1}$$

where $Q_{ik}\,Q_{jk} = \delta_{ij}$ for all time t.

Two interpretations may be given to this equation [33]. Firstly it may be thought to relate the same motion of a body as seen by observers in two different reference frames, $\tilde{S}$ and S, moving relative to one another. For this interpretation the transformation $Q_{ij}(t)$ belongs to the full orthogonal group, that is transformations for which $\det[Q_{ij}] = -1$ are included.

Equation (1.5.1) may also be interpreted as relating two motions of a body, differing only by a rigid body motion, as seen by a single observer. To be meaningful, $Q_{ij}$ must then belong to the group of proper orthogonal transformations, that is $\det[Q_{ij}] = +1$.

Returning to the first interpretation of equation (1.5.1), the following definition is useful. Quantities whose transformation from reference frame S to $\tilde{S}$ depends only on the relative orientation of the two reference frames and not on any other aspect of their relative motion are said to be frame-indifferent or objective. Thus frame-indifferent scalars, vectors, and tensors transform as

$$\tilde{A} = A$$

$$\tilde{v}_i = Q_{ij}(t)\, v_j \ ,$$

and

$$\tilde{b}_{ij} = Q_{ik}(t)\, Q_{jl}(t)\, b_{kl} \ ,$$

respectively for all time t.

Since the response of a material to some loading system must appear the same to all observers regardless of their motion, constitutive equations must be form invariant under a change of reference frame. Thus in general if the constitutive equation for a material as observed from reference frame S is of the form

$$\sigma_{ij} = H_{ij}(x_k(X_K,t)) \ ,$$

where the tensor components $H_{ij}$ are functionals of the entire deformation history of the body, then the constitutive equation for the same material as observed from the reference frame $\tilde{S}$ must be

$$\tilde{\sigma}_{ij} = H_{ij}(\tilde{x}_k(X_K,t)) \ ,$$

where $\tilde{\sigma}_{ij} = Q_{ik}(t)\ Q_{jl}(t)\ \sigma_{kl}$ and $\tilde{x}_k(X_K,t)$ is given by equation (1.5.1). This same result follows if the second interpretation of equation (1.5.1) is used.

It will now be shown that the Prandtl-Reuss equations given in the previous section do not satisfy the principle of material indifference. The results one obtains using the Prandtl-Reuss equations for any particular problem must be considered in view of this since the use of these equations in situations where rotations are significant may result in unacceptable errors.

Taking the total time derivative of equation (1.5.1) gives

$$\tilde{v}_i = \dot{c}_i(t) + Q_{ij}(t)\, v_j + \dot{Q}_{ij}(t)\, x_j \ .$$

Defining the antisymmetric tensor $\Omega_{ij}$ by

$$\Omega_{ij} = \dot{Q}_{ik}\, Q_{jk} = - \Omega_{ji} \qquad (1.5.2a,b)$$

the velocity gradient tensor observed from reference frame $\tilde{S}$ is

$$\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j} = Q_{ik}\, Q_{jl}\, \frac{\partial v_k}{\partial x_l} + \Omega_{ij} \ . \qquad (1.5.3)$$

Forming the symmetric and antisymmetric parts of the velocity gradient tensor gives

$$\tilde{d}_{ij} = Q_{ik}\, Q_{jl}\, d_{kl} \ ,$$

and

$$\tilde{\omega}_{ij} = Q_{ik}\, Q_{jl}\, \omega_{kl} + \Omega_{ij} \ ,$$

which are the stretching or rate of deformation and the spin or vorticity tensors respectively. Thus the rate of deformation tensor is frame-indifferent and the spin tensor is not.

Consider the total time derivative of the stress tensor

$$\tilde{\sigma}_{ij} = Q_{ik}\, Q_{jl}\, \sigma_{kl} \ .$$

This leads to

$$\dot{\tilde{\sigma}}_{ij} = \dot{Q}_{ik}Q_{jl}\sigma_{kl} + Q_{ik}\dot{Q}_{jl}\sigma_{kl} + Q_{ik}Q_{jl}\dot{\sigma}_{kl} , \tag{1.5.4}$$

which shows that the material derivative of the stress tensor, and hence the stress deviator tensor, is not frame-indifferent and thus the Prandtl-Reuss equations (1.4.3) are not form invariant under a change in reference frame. If the material derivative of stress in equation (1.4.2) was replaced by a frame-indifferent stress rate then it follows, since $d_{ij}$ and $\sigma'_{ij}$ are frame-indifferent, that equation (1.4.2) would be form invariant under changes of reference frame.

From equations (1.5.2) and (1.5.3) it is found that

$$\dot{Q}_{ik} = \frac{\partial \tilde{v}_i}{\partial \tilde{x}_r} Q_{rk} - Q_{ir} \frac{\partial v_r}{\partial x_k} \tag{1.5.5}$$

$$= - \frac{\partial \tilde{v}_r}{\partial \tilde{x}_i} Q_{rk} + Q_{ir} \frac{\partial v_k}{\partial x_r} . \tag{1.5.6}$$

Substituting equations (1.5.5) and (1.5.6) into equation (1.5.4) gives

$$\dot{\tilde{\sigma}}_{ij} - \tilde{\sigma}_{rj} \frac{\partial \tilde{v}_i}{\partial \tilde{x}_r} - \tilde{\sigma}_{ir} \frac{\partial \tilde{v}_j}{\partial \tilde{x}_r} = Q_{ik}Q_{jl}(\dot{\sigma}_{kl} - \sigma_{rl} \frac{\partial v_k}{\partial x_r} - \sigma_{kr} \frac{\partial v_l}{\partial x_r}) , \tag{1.5.7}$$

and

$$\dot{\tilde{\sigma}}_{ij} + \tilde{\sigma}_{rj} \frac{\partial \tilde{v}_r}{\partial \tilde{x}_i} + \tilde{\sigma}_{ir} \frac{\partial \tilde{v}_r}{\partial \tilde{x}_j} = Q_{ik}Q_{jl}(\dot{\sigma}_{kl} + \sigma_{rl} \frac{\partial v_r}{\partial x_k} + \sigma_{kr} \frac{\partial v_r}{\partial x_l}) , \tag{1.5.8}$$

respectively. Addition of equations (1.5.7) and (1.5.8) and use of the definition of the spin tensor gives

$$\dot{\tilde{\sigma}}_{ij} + \tilde{\omega}_{ri}\tilde{\sigma}_{rj} + \tilde{\omega}_{rj}\tilde{\sigma}_{ir} = Q_{ik}Q_{jl}(\dot{\sigma}_{kl}+\omega_{rk}\sigma_{rl}+\omega_{rl}\sigma_{kr}) \ . \qquad (1.5.9)$$

Using the notation

$$\frac{\delta\sigma_{ij}}{\delta t} = \dot{\sigma}_{ij} - \sigma_{rj}\frac{\partial v_i}{\partial x_r} - \sigma_{ir}\frac{\partial v_j}{\partial x_r} \ ,$$

$$\frac{\mathcal{D}}{\mathcal{D}t}\sigma_{ij} = \dot{\sigma}_{ij} + \sigma_{rj}\frac{\partial v_r}{\partial x_i} + \sigma_{ir}\frac{\partial v_r}{\partial x_j} \ ,$$

and

$$\hat{\sigma}_{ij} = \dot{\sigma}_{ij} + \sigma_{rj}\omega_{ri} + \sigma_{ir}\omega_{rj} \ ,$$

equations (1.5.7), (1.5.8), and (1.5.9) become

$$\frac{\delta\sigma_{ij}}{\delta t} = Q_{ik}Q_{jl}\frac{\delta}{\delta t}\sigma_{kl} \ ,$$

$$\frac{\mathcal{D}}{\mathcal{D}t}\tilde{\sigma}_{ij} = Q_{ik}Q_{jl}\frac{\mathcal{D}}{\mathcal{D}t}\sigma_{kl} \ ,$$

and

$$\hat{\tilde{\sigma}}_{ij} = Q_{ik}Q_{jl}\hat{\sigma}_{kl} \ .$$

This indicates that these stress rates, the Oldroyd [36], the Cotter-Rivlin [37], and the Jaumann [38] stress rates respectively, are all

frame-indifferent and may therefore be used in constitutive equations.

Other stress rates may be formed from any of the above stress rates by addition of terms which are frame-indifferent.

## 1.6 The Kinematically Correct Form of the Prandtl-Reuss Equations

As shown in the previous section the use of the material derivative of the stress or stress deviator tensor in constitutive equations violates the principle of material indifference. Thus one of the frame-indifferent stress rates must be used instead.

It has been suggested by Prager [39] that the most suitable definition of stress rate for the theory of plasticity is the Jaumann derivative, since only it has the property that its vanishing implies that the stress invariants are stationary. Since the yield condition for an isotropic material depends on the invariants of the stress deviator and since any observer who observes zero rate of change of stress should observe no change in the value of the yield function, the choice of the Jaumann stress rate seems justified.

The following physical interpretation may be given to the Jaumann derivative which adds to its desirability for use in constitutive equations. The Jaumann derivative of the stress tensor at a point in a deforming medium is the material derivative of the stress referred to a Cartesian coordinate system which instantaneously coincides with the fixed coordinate system but which participates in the rotation of the material particle at that point. Such a coordinate system has been called a kinematically preferred coordinate system by Thomas [40]

and the Jaumann derivative is sometimes referred to as the co-rotational derivative.

The kinematically correct or frame-indifferent form of the Prandtl-Reuss equations using the Jaumann stress rate were first developed by Thomas [20].

After substitution of the Jaumann stress rate for the material derivative of stress, the Prandtl-Reuss equations become

$$d_{ij} = \frac{1}{2\mu}\hat{\sigma}'_{ij} - \frac{1}{3K}\overset{\circ}{p}\,\delta_{ij} + \lambda\sigma'_{ij} ,$$

since

$$\hat{p} = \overset{\circ}{p} .$$

In particular for an incompressible material

$$d_{ij} = \frac{1}{2\mu}\hat{\sigma}'_{ij} + \lambda\sigma'_{ij} . \qquad (1.6.1)$$

This form of the Prandtl-Reuss equations is valid for large total deformations since the use of frame-indifferent tensors allows for finite rotations of the material elements. They are valid however, only for small elastic strains, that is elastic strains which may be described by the small strain tensor (1.2.1).

The constitutive equations (1.6.1) have been used by Haddow and Danyluk [41], [42], [43] to solve the problems of the plane and axisymmetric flow of an incompressible elastic-perfectly plastic material in a converging channel.

# CHAPTER II

# KINEMATICS FOR A LARGE STRAIN ELASTIC-PLASTIC THEORY

## 2.1 Introduction

In this chapter the kinematics of a large strain elastic-plastic theory are discussed and frame-indifferent tensor quantities are developed which may be used in constitutive equations without violating the principle of material indifference. The only assumption made is that both finite elastic and plastic strains may be possible in the materials considered.

Three configurations of a body composed of an elastic-plastic material are defined. These are the initial unstrained and unstressed configuration of the body at a uniform reference temperature, the unstressed plastically strained configuration of the body at the reference temperature, and the stressed configuration of the body in which elastic and possibly plastic strains may be present due to external loads and body forces. These configurations of the body will be referred to as C.1, C.2, and C.3 respectively. If no plastic strains have occurred then C.1 and C.2 differ at most by a rigid body motion. The use of three configurations to consider finite elastic-plastic problems has been adopted by Sedov [21], Backman [44], and recently by Lee [27].

If the deformation from C.1 to C.3 is non-homogeneous any

plastic strains which result will in general not satisfy the compatibility conditions for finite strain [45]. Thus residual stresses will result from removal of the external loads and body forces required to maintain the body in C.3 and the space corresponding to C.2 will then be non-Euclidean and C.2 will not be a configuration which the body may occupy physically. The configuration C.2 is then thought of as the configuration of the body which results from isolating each material element from the surrounding material and unloading it elastically and then returning it to the reference temperature.

A fourth configuration of the body C.4, which is used in Chapter VII results from removing all external loads and body forces from C.3 and returning the body to the uniform reference temperature. If the plastic strains satisfy the compatibility conditions then C.2 and C.4 coincide, otherwise residual stresses are present in C.4.

## 2.2 Deformation Gradients and Strain Tensors

Let the coordinates of a particle in C.1, C.2, and C.3, referred to a fixed Cartesian coordinate system, be $X_K$, $x_\alpha$, and $x_j$ respectively. Henceforth lower case Latin subscripts refer to C.3, lower case Greek subscripts to C.2, and upper case Latin subscripts to C.1.

The motion of the body is given by

$$x_k = x_k(X_K,t) \ , \qquad (2.2.1)$$

or $$x_k = x_k(\chi_\alpha,t) \text{ and } \chi_\alpha = \chi_\alpha(X_K,t) , \qquad (2.2.2a,b)$$

where in general the functions $x_k(X_K,t)$ and $x_k(\chi_\alpha,t)$ have different forms.

Continuity requires that the mapping (2.2.1) from C.1 to C.3 have a unique inverse in the neighbourhood of any given particle. It follows from the implicit function theorem [46] that $x_k(X_K,t)$ is continuous and possesses first order partial derivatives with respect to $X_K$ in the neighbourhood of a particle and that det $(\partial x_k/\partial X_K)$ does not vanish in that neighbourhood. These conditions are not necessarily true for the mappings from C.1 to C.2 and C.2 to C.3 since C.2 may not be a physically attainable configuration of the body.

Let dS, $d\mathcal{s}$, and ds be an infinitesimal element of length in C.1, C.2, and C.3 with components $dX_K$, $d\chi_\alpha$, and $dx_i$ respectively, referred to a fixed Cartesian coordinate system. Then

$$dx_i = F_{iK}dX_K = F^{(e)}_{i\alpha}d\chi_\alpha \qquad (2.2.3a,b)$$

and $$d\chi_\alpha = F^{(p)}_{\alpha K}dX_K \qquad (2.2.4)$$

where $F_{iK} = \frac{\partial x_i}{\partial X_K}$ is the total deformation gradient and $F^{(e)}_{i\alpha}$ and $F^{(p)}_{\alpha K}$ are the elastic and plastic deformation gradient tensors respectively. These tensors are related by

$$F_{iK} = F^{(e)}_{i\alpha} F^{(p)}_{\alpha K} .$$

If the space corresponding to C.2 is Euclidean then

$$F^{(e)}_{i\alpha} = \frac{\partial x_i}{\partial \chi_\alpha}$$

and

$$F^{(p)}_{\alpha K} = \frac{\partial \chi_\alpha}{\partial X_K} .$$

If this space is non-Euclidean the above partial derivatives do not exist but equations (2.2.3) and (2.2.4) are still valid with the matrices $F^{(e)}_{i\alpha}$ and $F^{(p)}_{\alpha K}$ representing local linear mappings. Although the remainder of the discussion in this chapter is restricted to configurations C.2 which are Euclidean, the results remain valid in general provided velocity and deformation gradients with respect to C.2 are not expressed as partial derivatives.

The assumption from classical plasticity that the plastic volume change is zero is retained here and thus

$$\det\left(\frac{\partial \chi_\alpha}{\partial X_K}\right) = 1 . \tag{2.2.5}$$

Furthermore

$$\det\left(\frac{\partial x_i}{\partial \chi_\alpha}\right) = \det\left(\frac{\partial x_i}{\partial X_K}\right) = \frac{\rho_1}{\rho_3} ,$$

where $\rho_1$ and $\rho_3$ are the densities in C.1 and C.3 respectively.

From

$$dS^2 = dX_K dX_K ,$$

$$ds^2 = d\chi_\alpha d\chi_\alpha ,$$

and

$$ds^2 = dx_i dx_i ,$$

and equations (2.2.3a,b) and (2.2.4) the following results are obtained,

$$ds^2 - dS^2 = 2E_{KL} dX_K dX_L ,$$

$$ds^2 - ds^2 = 2E^{(e)}_{\alpha\beta} d\chi_\alpha d\chi_\beta ,$$

and

$$ds^2 - dS^2 = 2E^{(p)}_{KL} dX_K dX_L ,$$

where

$$E_{KL} = \frac{1}{2} \left(\frac{\partial x_i}{\partial X_K} \frac{\partial x_i}{\partial X_L} - \delta_{KL}\right) ,$$

$$E^{(e)}_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial x_i}{\partial \chi_\alpha} \frac{\partial x_i}{\partial \chi_\beta} - \delta_{\alpha\beta}\right) ,$$

and

$$E^{(P)}_{KL} = \frac{1}{2} \left(\frac{\partial \chi_\alpha}{\partial X_K} \frac{\partial \chi_\alpha}{\partial X_L} - \delta_{KL}\right) , \qquad (2.2.6a,b,c)$$

are the total, the elastic, and the plastic Lagrangian strain tensors

referred to C.1, C.2, and C.1 respectively.

## 2.3 Strain Rate and Rate of Deformation Tensors

The material derivative of equation (2.2.6a), that is the time derivative holding $X_K$ constant, is

$$\dot{E}_{KL} = d_{ij} \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_L} , \tag{2.3.1}$$

where
$$d_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2.3.2}$$

is the total deformation rate tensor and $v_i = \dot{x}_i$ is the velocity of a particle in C.3.

Similarly material differentiation of $E_{KL}^{(P)}$ gives

$$\dot{E}_{KL}^{(P)} = d_{\alpha\beta}^{(P)} \frac{\partial \chi_\alpha}{\partial X_K} \frac{\partial \chi_\beta}{\partial X_L} ,$$

where
$$d_{\alpha\beta}^{(P)} = \frac{1}{2} \left(\frac{\partial \nu_\alpha}{\partial \chi_\beta} + \frac{\partial \nu_\beta}{\partial \chi_\alpha}\right)$$

is a plastic rate of deformation tensor and $\nu_\alpha = \dot{\chi}_\alpha$ is the velocity of a particle in C.2. Equation (2.2.5) which follows from the assumption of zero plastic volume change implies that

$$d_{\alpha\alpha}^{(P)} = 0 .$$

The material derivative of the equation (2.2.6b) is*

$$\dot{E}^{(e)}_{\alpha\beta} = d_{ij}\frac{\partial x_i}{\partial \chi_\alpha}\frac{\partial x_j}{\partial \chi_\beta} - \frac{1}{2}\frac{\partial \nu_\gamma}{\partial \chi_\alpha}\frac{\partial x_i}{\partial \chi_\gamma}\frac{\partial x_i}{\partial \chi_\beta} - \frac{1}{2}\frac{\partial \nu_\gamma}{\partial \chi_\beta}\frac{\partial x_i}{\partial \chi_\gamma}\frac{\partial x_i}{\partial \chi_\alpha} . \qquad (2.3.3)$$

The velocity gradient $\partial\nu_\alpha/\partial\chi_\beta$ which appears in equation (2.3.3) may be written as

$$\frac{\partial \nu_\alpha}{\partial \chi_\beta} = d^{(P)}_{\alpha\beta} + \omega^{(P)}_{\alpha\beta} , \qquad (2.3.4)$$

where $$\omega^{(P)}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \nu_\alpha}{\partial \chi_\beta} - \frac{\partial \nu_\beta}{\partial \chi_\alpha}\right)$$

is the plastic spin tensor. Configuration C.2 is defined only to within a rigid body rotation and the plastic spin tensor $\omega^{(P)}_{\alpha\beta}$ is not uniquely defined. The strain rate $\dot{E}^{(e)}_{\alpha\beta}$ depends on $\omega^{(P)}_{\alpha\beta}$ and consequently it is an unsatisfactory definition of elastic strain rate. In fact it can have non-zero components when the body is undergoing a rigid body motion with neither elastic nor plastic deformation occurring. The necessary and sufficient condition for a rigid body motion is

$$d_{ij} \equiv 0$$

throughout the body and the condition for no plastic deformation is

$$d^{(P)}_{\alpha\beta} \equiv 0$$

*In taking this material derivative it is noted that the coordinates $\chi_\alpha$ are functions of time.

throughout the body. If both conditions are met then no elastic deformation is occurring but $\omega_{\alpha\beta}^{(P)}$ is in general not zero for a rigid body motion of C.2 and it can be deduced from equations (2.3.3) and (2.3.4) that $\dot{E}_{\alpha\beta}^{(e)}$ is non-zero when $d_{ij} \equiv 0$ and $d_{ij}^{(P)} \equiv 0$ if $\omega_{\alpha\beta}^{(P)} \neq 0$. An elastic strain rate tensor which does not depend on $\omega_{\alpha\beta}^{(P)}$ is obtained as follows.

The plastic deformation gradient may be written as

$$\frac{\partial x_\alpha}{\partial X_K} = R_{\alpha L}^{(P)} U_{LK}^{(P)} ,$$

where $R_{\alpha L}^{(P)}$ is a proper orthogonal tensor and $U_{LK}^{(P)}$ is a symmetric positive definite tensor such that

$$U_{KM}^{(P)} U_{LM}^{(P)} = \frac{\partial x_\alpha}{\partial X_K} \frac{\partial x_\alpha}{\partial X_L} .$$

The deformation from C.1 to C.2 consists of a pure stretch deformation given by $U_{KL}^{(P)}$ followed by a rigid body rotation given by $R_{\alpha K}^{(P)}$. Therefore unlike $E_{\alpha\beta}^{(e)}$, the tensor

$$E_{KL}^{(e)} = R_{\alpha K}^{(P)} R_{\beta L}^{(P)} E_{\alpha\beta}^{(e)} \tag{2.3.5}$$

is independent of any rigid body rotation of C.2.

The material derivative of equation (2.3.5) is

$$\overset{\circ}{E}{}^{(e)}_{KL} = R^{(P)}_{\alpha K} R^{(P)}_{\beta L} \hat{E}^{(e)}_{\alpha\beta} + Q_{KM} E^{(e)}_{ML} - E^{(e)}_{KM} Q_{ML} , \qquad (2.3.6)$$

where

$$Q_{KL} = \frac{1}{2} (\dot{U}^{(P)}_{KM} U^{(P)^{-1}}_{ML} - U^{(P)^{-1}}_{KM} \overset{\circ}{U}{}^{(P)}_{ML}) \qquad (2.3.7)$$

is an antisymmetric tensor and

$$\hat{E}^{(e)}_{\alpha\beta} = \dot{E}^{(e)}_{\alpha\beta} + E^{(e)}_{\alpha\gamma} \omega^{(P)}_{\gamma\beta} - \omega^{(P)}_{\alpha\gamma} E^{(e)}_{\gamma\beta} \qquad (2.3.8)$$

is the co-rotational derivative of $E^{(e)}_{\alpha\beta}$, that is the material derivative referred to a Cartesian coordinate system which participates in the rotation of the material in C.2 at the point considered and which instantaneously coincides with the fixed coordinate system. Thus $\hat{E}^{(e)}_{\alpha\beta}$ is independent of the spin of C.2.

Using equations (2.3.3) and (2.3.4) the co-rotational derivative of $E^{(e)}_{\alpha\beta}$ becomes

$$\begin{aligned} \hat{E}^{(e)}_{\alpha\beta} &= \dot{E}^{(e)}_{\alpha\beta} + \frac{1}{2} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_i}{\partial X_\gamma} \omega^{(P)}_{\gamma\beta} + \frac{1}{2} \frac{\partial x_i}{\partial X_\gamma} \frac{\partial x_i}{\partial X_\beta} \omega^{(P)}_{\gamma\alpha} \\ &= d_{ij} \frac{\partial x_i}{\partial X_\alpha} \frac{\partial x_j}{\partial X_\beta} - \frac{1}{2} d^{(P)}_{\gamma\alpha} \frac{\partial x_i}{\partial X_\gamma} \frac{\partial x_i}{\partial X_\beta} - \frac{1}{2} d^{(P)}_{\gamma\beta} \frac{\partial x_i}{\partial X_\gamma} \frac{\partial x_i}{\partial X_\alpha} . \end{aligned} \qquad (2.3.9)$$

An elastic rate of deformation tensor which is independent of the spin of C.2 is given by

$$d_{ij}^{(e)} = \frac{\partial \chi_\alpha}{\partial x_i} \frac{\partial \chi_\beta}{\partial x_j} \hat{E}_{\alpha\beta}^{(e)} ,$$

and from equation (2.3.9)

$$d_{ij}^{(e)} = d_{ij} - d_{ij}^* \tag{2.3.10}$$

where
$$d_{ij}^* = \frac{1}{2} \left(\frac{\partial \chi_\beta}{\partial x_i} d_{\alpha\beta}^{(P)} \frac{\partial x_j}{\partial \chi_\alpha} + \frac{\partial \chi_\beta}{\partial x_j} d_{\alpha\beta}^{(P)} \frac{\partial x_i}{\partial \chi_\alpha}\right) . \tag{2.3.11}$$

The tensor $d_{ij}^*$ is independent of the rotation of C.2 and the spin of C.2 and C.3. It has the property that $d_{kk}^* = 0$ for elastic-plastic flow with no plastic volume change. Thus $d_{ij}^*$ is a possible definition for plastic rate of deformation and it is shown in section 3.7 that it is a suitable definition for plastic rate of deformation for an elastic-plastic material which is elastically isotropic.

# CHAPTER III

## HYPERELASTICITY AND THERMOELASTIC CONSIDERATIONS FOR AN ELASTIC-PLASTIC MATERIAL

### 3.1 Introduction

The constitutive equations (1.2.2) for the classical Hookean elastic material are based on a rather restricted definition of elastic behavior. In fact Hookean elastic behavior is a special case of a much broader concept of elasticity; the existence of a stress free natural state and the unique dependence of the stress on the deformation gradients relative to the natural state and on temperature. An elastic material possesses a perfect memory of the natural state. In this chapter elastic materials [47] which possess an elastic potential or strain energy function are considered and these materials are described as hyperelastic.

### 3.2 Kinematics

Let C.1 be the undeformed configuration or natural state of an elastic body. To be consistent with the notation introduced in Chapter II the deformed configuration of the body is called C.3 and since the body is purely elastic no reference is made here to configurations C.2 and C.4 discussed previously in connection with elastic-plastic materials.

Let $\xi_1$, $\xi_2$, and $\xi_3$ be the coordinates of a material point referred to a curvilinear convected coordinate system [48], that is a coordinate system whose coordinate surfaces consist of the same material particles at all times during the deformation. Let the coordinates of a particle referred to a fixed rectangular Cartesian coordinate system be $X_K$ and $x_k$ in C.1 and C.3 respectively. The convected coordinate system is defined by

$$X_K = X_K\,(\xi_1,\xi_2,\xi_3)\ , \qquad (3.2.1)$$

where $X_K$ are single valued, continuous functions of the $\xi_i$, and the motion of the body is given by

$$x_i = x_i(\xi_1,\xi_2,\xi_3,t)\ , \qquad (3.2.2)$$

where the $x_i$ are continuous and single valued functions of $\xi_i$ for any fixed t.

If $\overline{r}$ is the position vector of a particle P in C.3 referred to the origin of the fixed Cartesian coordinate system and $\overline{R}$ is the position vector of the same particle in C.1, the displacement vector is

$$\overline{u} = \overline{r} - \overline{R}\ . \qquad (3.2.3)$$

From equations (3.2.1) and (3.2.2) it follows that

$$\overline{R} = \overline{R}(\xi_1, \xi_2, \xi_3) ,$$

$$\overline{r} = \overline{r}(\xi_1, \xi_2, \xi_3, t) ,$$

and

$$\overline{u} = \overline{u}(\xi_1, \xi_2, \xi_3, t) .$$

If the vectors $\overline{e}_i$ are unit vectors along the coordinate axes of the Cartesian coordinate system then*

$$\overline{R} = X^K \, \overline{e}_K ,$$

and

$$\overline{r} = x^k \, \overline{e}_k .$$

The local covariant bases vectors at an arbitrary point $P_o$ in C.1 and P in C.3 for this curvilinear coordinate system are then

$$\overline{G}_i = \overline{R}_{,i} = \left(\frac{\partial X^K}{\partial \xi^i}\right)_{P_o} \overline{e}_K ,$$

and

$$\overline{g}_i = \overline{r}_{,i} = \left(\frac{\partial x^k}{\partial \xi^i}\right)_P \overline{e}_k \qquad (3.2.4a,b)$$

---

*The same Cartesian coordinate system is used to describe C.1 and C.3 so that $\overline{e}_k$ and $\overline{e}_K$ refer to the same vectors, however the lower and upper case subscripts are both used so as to be consistent with the summation convention.

respectively, where the comma followed by the subscript i denotes partial differentiation with respect to $\xi_i$ and the subscripts on the parentheses denote the points at which the enclosed quantities are evaluated.

The contravariant base vectors in C.1 and C.3 are defined by the inner products

$$\bar{G}^i \cdot \bar{G}_j = \delta^i{}_j \ ,$$

and

$$\bar{g}^i \cdot \bar{g}_j = \delta^i{}_j \ .$$

Thus the components of the metric tensors in C.1 and C.3, referred to the convected coordinate system are

$$G^{ij} = \frac{\partial \xi^i}{\partial X^K} \frac{\partial \xi^j}{\partial X^K} \ , \ G_{ij} = \frac{\partial X^K}{\partial \xi^i} \frac{\partial X^K}{\partial \xi^j} \ , \qquad (3.2.5a,b)$$

and

$$g^{ij} = \frac{\partial \xi^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^k} \ , \ g_{ij} = \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^k}{\partial \xi^j} \qquad (3.2.6a,b)$$

so that the lengths dS and ds of an infinitesimal line element in C.1 and C.3 are given by

$$dS^2 = G_{ij} d\xi^i d\xi^j \ ,$$

and

$$ds^2 = g_{ij} d\xi^i d\xi^j \ .$$

The difference of the squares of the lengths is

$$ds^2 - dS^2 = \gamma_{ij} d\xi^i d\xi^j \ ,$$

where
$$\gamma_{ij} = \frac{1}{2} (g_{ij} - G_{ij}) \tag{3.2.7}$$

is a strain tensor. From equations (3.2.5b), (3.2.6b), (2.2.6a), and (3.2.7) it follows that

$$\gamma_{ij} = E_{KL} \frac{\partial X^K}{\partial \xi^i} \frac{\partial X^L}{\partial \xi^j} \tag{3.2.8}$$

and from equations (3.2.3), (3.2.4), and (3.2.7)

$$\gamma_{ij} = \frac{1}{2} [(\overline{G}_i + \overline{u}_{,i}) \cdot (\overline{G}_j + \overline{u}_{,j}) - G_{ij}] \ .$$

The material derivative of this equation is

$$\dot{\gamma}_{ij} = \frac{1}{2} (v_{j;i} + v_{i;j}) \ , \tag{3.2.9}$$

where $v_i$ are the covariant components of the velocity vector in C.3 referred to the local basis $\overline{g}^i$ and a subscript following a semi-colon indicates covariant differentiation with respect to $\xi^i$ in C.3. That is

$$v_{i;j} = v_{i,j} - \Gamma^k_{ij} v_k \ ,$$

where the Christoffel symbols $\Gamma^k_{ij}$ are evaluated from the metric tensor $g_{ij}$.

The components of $\dot{\gamma}_{ij}$ in equation (3.2.9) are the components of the rate of deformation tensor $d_{ij}$ referred to the convected co-ordinate system.

Mixed tensors may be formed from $\gamma_{ij}$ in two ways by using the metric tensors from C.1 and C.3 to raise one index. The following definition is adopted

$$\gamma^i{}_j = G^{ik}\gamma_{kj}$$

$$= \frac{1}{2}\,(G^{ik}g_{kj} - \delta^i{}_j)\ .$$

Three independent invariants of $\gamma^i{}_j$ are of interest. They are also the principal invariants of Green's deformation tensor [49]

$$C_{KL} = \frac{\partial x_i}{\partial X_K}\,\frac{\partial x_i}{\partial X_L}$$

and Finger's deformation tensor [50]

$$B_{ij} = \frac{\partial x_i}{\partial X_K}\,\frac{\partial x_j}{\partial X_K}$$

and are the coefficients of the powers of $\eta$ in the expansion of the determinant

$$|\eta\delta^i{}_j + \delta^i{}_j + 2\gamma^i{}_j| = |\eta\delta^i{}_j + G^{ik}g_{kj}|$$

$$= \eta^3 + I_1\eta^2 + I_2\eta + I_3 = 0 \ .$$

It follows that

$$I_1 = 3 + 2\gamma^k{}_k = G^{ij}g_{ij} \ ,$$

$$I_2 = 3 + 4\gamma^k{}_k + 2(\gamma^k{}_k\gamma^m{}_m - \gamma^k{}_m\gamma^m{}_k)$$

$$= \frac{1}{2}\,(I_1{}^2 - G^{im}G^{jn}g_{ij}g_{mn}) \ ,$$

and
$$I_3 = \det(G^{im}g_{mj}) = g/G \ ,$$

where
$$G = \det(G_{ij}) \text{ and } g = \det(g_{ij}) \ .$$

## 3.3 Stress Tensor and Equations of Motion

The Cauchy stress tensor, which is symmetric, has components $\sigma_{ij}$ referred to a Cartesian coordinate system. Consider the contravariant and mixed components of this tensor referred to the convected coordinate system in C.3. From the transformation law for second order tensors these components are

$$\tau^{rs} = \frac{\partial\xi^r}{\partial x^i}\frac{\partial\xi^s}{\partial x^j}\,\sigma_{ij} \ , \tag{3.3.1}$$

and
$$\tau^r{}_s = \frac{\partial \xi^r}{\partial x^i} \frac{\partial x^j}{\partial \xi^s} \sigma_{ij} \ .$$

It follows that $\tau^{rs}$ is symmetric.

If $\overline{T}$ is the stress vector acting on a surface in C.3 with unit normal vector $\overline{n}$ where

$$\overline{n} = n_i \overline{e}_i = n_j \overline{g}^j$$

then

$$\overline{T} = \sigma_{ij} n_i \overline{e}_j = \tau^r{}_s n_r \overline{g}^s \ .$$

Cauchy's first law applied to an arbitrary material volume $V$ with closed surface $S$ in C.3 is

$$\int_S \overline{T} \, dS + \int_V \rho_3 (\overline{b} - \overline{a}) dV = 0 \ , \qquad (3.3.2)$$

where $\overline{b}$ is the body force vector per unit mass, $\overline{a}$ is the acceleration vector and $\rho_3$ the density in C.3. Equation (3.3.2) is an analogue of Newton's second law and may be derived from an energy balance on the volume $V$ using invariance requirements under rigid body motions [51].

Applying Greens theorem [52] to the surface integral in equation (3.3.2) gives

$$\int_V [\tau^r{}_{s;r} + \rho_3 (b_s - a_s)] \overline{g}^s \, dV = 0 \ .$$

Since the volume $V$ is arbitrary and the vectors $\bar{g}^s$ form a basis in C.3, Cauchy's first equation of motion in terms of components referred to the basis $\bar{g}^s$ in C.3 is

$$\tau^r{}_{s;r} + \rho_3(b_s - a_s) = 0 \ . \tag{3.3.3}$$

## 3.4 The Constitutive Equations for an Hyperelastic Material

The rate at which the surface forces do work on $S$ plus the rate at which the body forces do work on the mass in the volume $V$ minus the rate of increase of the kinetic energy of the mass in $V$ is

$$H = \int_S \bar{T} \cdot \bar{v} dS + \int_V \rho_3 \bar{b} \cdot \bar{v} dV - \int_V \rho_3 \bar{a} \cdot \bar{v} dV \tag{3.4.1}$$

$$= \int_S \tau^r{}_s n_r v^s dS + \int_V \rho_3 (b_s - a_s) v^s dV .$$

Converting the surface integral into a volume integral yields, after rearrangement

$$H = \int_V [(\tau^r{}_{s;r} + \rho_3 b_s - \rho_3 a_s) v^s + \tau^r{}_s v^s{}_{;r}] \, dV .$$

The first term in the integrand is zero by equation (3.3.3) so that

$$H = \int_V \tau^r{}_s v^s{}_{;r} dV = \int_V \tau^{rs} v_{s;r} dV \ ,$$

and from the symmetry of the stress tensor

$$H = \frac{1}{2} \int_V \tau^{rs}(v_{s;r} + v_{r;s})\, dV \ ,$$

or using equation (3.2.9)

$$H = \int_V \tau^{rs} \overset{\circ}{\gamma}_{rs}\, dV \ .$$

Also from equations (2.3.1), (3.2.8), and (3.3.1) this may be written as

$$H = \int_V \sigma_{ij} d_{ij} dV \ .$$

Materials for which the stress $\tau^{rs}$ at a point is a single valued function of the deformation at that point, as given by $\gamma_{ij}$, and for which

$$\tau^{rs} \overset{\circ}{\gamma}_{rs} = \rho_3 \overset{\circ}{E} \ , \tag{3.4.2}$$

where E at a point is a continuous single valued differentiable function of $\gamma_{ij}$, are called hyperelastic materials and the function E is called the elastic potential or strain energy function per unit mass.

Since $$E = E(\gamma_{ij}) \ ,$$

it follows that

$$\dot{E} = \frac{\partial E}{\partial \gamma_{ij}} \dot{\gamma}_{ij} \quad , \tag{3.4.3}$$

where all nine components of $\gamma_{ij}$ are taken to be independent for the purpose of partial differentiation. From equations (3.4.2) and (3.4.3) the stress tensor is

$$\tau^{rs} = \rho_3 \frac{\partial E}{\partial \gamma_{rs}} \quad .$$

Expressed in terms of W, the elastic potential per unit volume in C.1, this becomes

$$\tau^{rs} = \frac{1}{\sqrt{I_3}} \frac{\partial W}{\partial \gamma_{rs}} \quad , \tag{3.4.4}$$

where $W = \rho_1 E$, and $\rho_1$ is the density in C.1.

In general for a homogeneous isotropic hyperelastic body, W is a function only of the invariants $I_1$, $I_2$, $I_3$, and it can be shown that [53]

$$\tau^{rs} = \Phi G^{rs} + \Psi\, b^{rs} + pg^{rs} \quad , \tag{3.4.5}$$

where $$\Phi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1} \quad ,$$

$$\Psi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2} ,$$

$$p = 2\sqrt{I_3} \frac{\partial W}{\partial I_3} , \qquad (3.4.6)$$

and $$b^{rs} = I_1 G^{rs} - G^{mr} G^{ns} g_{mn} .$$

The constitutive equation (3.4.5) referred to a fixed Cartesian coordinate system is

$$\sigma_{ij} = p\delta_{ij} + (\Phi + I_1\Psi)B_{ij} - \Psi B_{ik}B_{kj} , \qquad (3.4.7)$$

where $$B_{ij} = \frac{\partial x_i}{\partial X_M} \frac{\partial x_j}{\partial X_M}$$

are the components of Finger's strain tensor [50].

For an incompressible material

$$I_3 = 1 ,$$

and p, which represents an hydrostatic pressure, cannot be determined from equation (3.4.6) but must be found from the equilibrium equation and the boundary conditions for the particular problem.

Two particular forms of the elastic potential W for elastically incompressible materials are that postulated by Mooney [54] for which

$$W = C_1(I_1-3) + C_2(I_2-3) ,$$

and that for a neo-Hookean material for which

$$W = C(I_1-3) .$$

For small strains the constitutive equation for a neo-Hookean material reduces to that for an incompressible Hookean elastic material with a Young's modulus of 6C. A theoretical derivation of the form of W for a neo-Hookean material is shown in a book by Treloar [55] on the physics of rubber elasticity.

The above discussion does not include the effects of temperature or temperature gradients. A more general analysis in section 3.6 which considers thermal effects shows that for at least two particular types of loading an elastic potential W does exist.

### 3.5 Energy Balance and the Entropy Inequality

The principle of conservation of energy states that in a material volume $V$ with surface $S$ in configuration C.3, the rate of change $\dot{K}$ of the kinetic energy plus the rate of change $\dot{U}$ of the internal energy, must equal $P$ the sum of the rate at which the surface tractions do work on $S$ and the rate at which the body forces do work on the mass in $V$, plus the rate at which all other energy such as heat, electrical, or chemical energy is added to the volume $V$. In the present discussion only heat energy is considered in the last

term. This principle may then be written as

$$\dot{K} + \dot{U} = P + Q \tag{3.5.1}$$

where $Q$ is the rate at which heat is added to the volume $V$.

Rearranging equation (3.5.1) and recalling the definition of $H$ in equation (3.4.1) gives

$$\dot{U} = H + Q ,$$

or*

$$\frac{D}{Dt}\int_V \rho_3 U dV = \int_V \sigma_{ij} d_{ij} dV - \int_S q_i n_i dS + \int_V \rho_3 h \; dV , \tag{3.5.2}$$

where U is the internal energy per unit mass, $q_i$ are the components of the heat vector per unit area at a point on $S$ with unit normal $n_i$, and h is a heat source per unit mass in $V$.

Conservation of mass gives that

$$\frac{D}{Dt}\int_V \rho_3 \; dV = 0 ,$$

so that equation (3.5.2) becomes, after converting the surface integral

---

*A superposed dot and D/Dt both denote material differentiation.

to a volume integral,

$$\int_V \left(\rho_3 \dot{U} - \sigma_{ij} d_{ij} + \frac{\partial q_i}{\partial x_i} - \rho_3 h\right) dV$$

and since the volume $V$ is arbitrary, the energy equation becomes, in local form

$$\rho_3 \dot{U} - \sigma_{ij} d_{ij} + \frac{\partial q_i}{\partial x_i} - \rho_3 h = 0 \ . \tag{3.5.3}$$

Denoting by S the entropy per unit mass and the temperature by T, (T>0), the following entropy inequality is postulated

$$\frac{D}{Dt} \int_V \rho_3 S dV \geq \int_V \rho_3 \frac{h}{T} \, dV - \int_S \frac{q_i n_i}{T} \, dS \ .$$

This is known as the Clausius-Duhem inequality [56], and it states that the total rate of increase of entropy in the body is greater than or equal to the rate at which entropy flows through the surface $S$, plus the rate at which entropy is produced by sources within the body. The inequality simplifies to

$$\rho_3 \dot{S} \geq \rho_3 \frac{h}{T} - \frac{1}{T} \frac{\partial q_i}{\partial x_i} + \frac{q_i}{T^2} \frac{\partial T}{\partial x_i} \ ,$$

and using the energy equation (3.5.3) this becomes

$$\rho_3(T\dot{S} - \dot{U}) + \sigma_{ij}d_{ij} - \frac{q_i}{T}\frac{\partial T}{\partial x_i} \geq 0 \ . \qquad (3.5.4)$$

Expressed in terms of A, the Helmholtz free energy per unit mass [57] defined by

$$A = U - TS, \qquad (3.5.5)$$

equation (3.5.4) becomes

$$- \rho_3(\dot{A} + \dot{T}S) + \sigma_{ij}d_{ij} - \frac{q_i}{T}\frac{\partial T}{\partial x_i} \geq 0 \ . \qquad (3.5.6)$$

## 3.6 Thermoelasticity

A theory of elasticity based on the general definition of elasticity but which includes thermal effects is known as thermoelasticity [58]. The constitutive equations for materials included in this theory may be written as

$$\sigma_{ij} = \sigma_{ij}(T, \frac{\partial x_i}{\partial X_K}) \ ,$$

$$A = A(T, \frac{\partial x_i}{\partial X_K}) \ ,$$

$$S = S(T, \frac{\partial x_i}{\partial X_K}) \ ,$$

and

$$q_k = q_k \ (T, \frac{\partial T}{\partial x_j} \ , \frac{\partial x_i}{\partial X_K}) \ .$$

The principle of material indifference gives

$$A = A(T, E_{KL})$$

and $$S = S(T, E_{KL}) \ .$$

The material derivative of the free energy A using equation (2.3.1) is

$$\dot{A} = \frac{\partial A}{\partial T} \dot{T} + \frac{\partial A}{\partial E_{KL}} \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_L} d_{ij} \ .$$

Substitution into the entropy inequality (3.5.6) gives

$$- \rho_3 \left(\frac{\partial A}{\partial T} + S\right) \dot{T} + \left(\sigma_{ij} - \rho_3 \frac{\partial A}{\partial E_{KL}} \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_L}\right) d_{ij} - \frac{q_i}{T} \frac{\partial T}{\partial x_i} \geq 0 \ . \quad (3.6.1)$$

Since $\dot{T}$ and $d_{ij}$ may be chosen arbitrarily, with the restriction that $d_{ij}$ be symmetric, inequality (3.6.1) implies that

$$S = - \frac{\partial A}{\partial T} \ , \quad (3.6.2)$$

$$\sigma_{ij} = \rho_3 \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_L} \frac{\partial A}{\partial E_{KL}} \ , \quad (3.6.3)$$

and

$$- \frac{q_i}{T} \frac{\partial T}{\partial x_i} \geq 0 \ .$$

Considering the internal energy as a function of S and $E_{KL}$ the entropy inequality (3.5.4) implies that

$$T = \frac{\partial U}{\partial S} ,$$

and

$$\sigma_{ij} = \rho_3 \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_L} \frac{\partial U}{\partial E_{KL}} . \tag{3.6.4}$$

Equations (3.6.3) and (3.6.4) indicate that

$$\left(\frac{\partial A}{\partial E_{KL}}\right)_T = \left(\frac{\partial U}{\partial E_{KL}}\right)_S$$

where the subscripts on the brackets indicate variables which are held constant for the differentiation. The constitutive equations (3.6.3) and (3.6.4) are valid for any type of loading. In particular however, equation (3.4.4) with $W = \rho_1 A$ is a special case of equation (3.6.3) for an isothermal process and with $W = \rho_1 U$ it is a special case of equation (3.6.4) for an isentropic process.

## 3.7 Thermoelastic Considerations for an Elastic-Plastic Material

Consider an elastic-perfectly plastic body with the configurations C.1, C.2, and C.3 defined in section 2.1. For convenience the Piola-Kirchoff stress tensor [59] referred to C.2,

$$S_{\alpha\beta} = \frac{\rho_1}{\rho_3} \frac{\partial \chi_\alpha}{\partial x_i} \frac{\partial \chi_\beta}{\partial x_j} \sigma_{ij} , \tag{3.7.1}$$

and the stress tensor with components

$$S_{KL} = R^{(P)}_{\alpha K}\ R^{(P)}_{\beta L} S_{\alpha\beta}, \tag{3.7.2}$$

where the proper orthogonal tensor $R^{(P)}_{\alpha K}$ is defined in section 2.3, are introduced. The assumption from classical plasticity that the plastic volume change be zero is retained so that in equation (3.7.1) $\rho_1$ appears since it equals $\rho_2$.

Unlike $S_{\alpha\beta}$, the components of $S_{KL}$ remain unchanged during a rigid body motion of C.2. If it is assumed that the elastic moduli and specific heats are unchanged by previous plastic deformation, the result

$$S_{KL} = S_{KL}(T, E^{(e)}_{MN}) \tag{3.7.3}$$

is obtained since for unloading from the elastic-plastic state the body behaves as a hyperelastic material. Furthermore the constitutive assumptions

$$A = A(T, E^{(e)}_{KL}, E^{(P)}_{KL}) \tag{3.7.4}$$

and

$$S = S(T, E^{(e)}_{KL}, E^{(P)}_{KL})$$

are made.* Green and Naghdi [23] and Kestin [60] have assumed that the

*A and S might depend on the plastic strain history as well as on $E^{(P)}_{KL}$.

entropy is independent of the plastic strain, for the finite and small strain elastic-plastic theories respectively, however there seems to be no physical basis for this assumption.

For elastic unloading of the body from C.3 the entropy inequality (3.5.6) becomes

$$-(\dot{A}+\dot{T}S) + \frac{\sigma_{ij}d_{ij}^{(e)}}{\rho_3} - \frac{q_i}{\rho_3 T}\frac{\partial T}{\partial x_i} \geq 0 \qquad (3.7.5)$$

since $d^*_{ij} \equiv 0$. From the definitions of $S_{\alpha\beta}$ and $d_{ij}^{(e)}$ it follows that

$$\frac{1}{\rho_3}\sigma_{ij}d_{ij}^{(e)} = \frac{1}{\rho_1} S_{\alpha\beta}\,\hat{E}_{\alpha\beta}^{(e)}$$

and from equations (2.3.6) and (3.7.2) this becomes

$$\frac{1}{\rho_3}\sigma_{ij}d_{ij}^{(e)} = \frac{1}{\rho_1} S_{KL}\dot{E}_{KL}^{(e)}$$

since $Q_{KL} \equiv 0$ when there is no plastic deformation occurring. Equation (3.7.5) may then be written as

$$-(\dot{A}+\dot{T}S) + \frac{1}{\rho_1} S_{KL}\dot{E}_{KL}^{(e)} - \frac{q_i}{\rho_3 T}\frac{\partial T}{\partial x_i} \geq 0 \ .$$

From equation (3.7.4) it follows that for elastic unloading

$$\dot{A} = \frac{\partial A}{\partial T}\dot{T} + \frac{\partial A}{\partial E_{KL}^{(e)}}\dot{E}_{KL}^{(e)} \qquad (3.7.6)$$

so that the entropy inequality for unloading is, after substitution of (3.7.6)

$$-(\frac{\partial A}{\partial T} + S)\dot{T} + (-\frac{\partial A}{\partial E^{(e)}_{KL}} + \frac{1}{\rho_1} S_{KL})\dot{E}^{(e)}_{KL} - \frac{q_i}{\rho_3 T}\frac{\partial T}{\partial x_i} \geq 0 \ . \qquad (3.7.7)$$

Both $\dot{T}$ and $\dot{E}^{(e)}_{KL}$ may be chosen arbitrarily for elastic unloading or reloading before further yielding so that equation (3.7.7) gives

$$S = -\left(\frac{\partial A}{\partial T}\right)_{E^{(e)},E^{(P)}} \ , \qquad (3.7.8)$$

$$S_{KL} = \rho_1\left(\frac{\partial A}{\partial E^{(e)}_{KL}}\right)_{T,E^{(P)}} \ , \qquad (3.7.9)$$

and

$$-\frac{q_i}{\rho_3 T}\frac{\partial T}{\partial x_i} \geq 0 \ .$$

Equations (3.7.8) and (3.7.9) give the Maxwell relation

$$-\rho_1\left(\frac{\partial S}{\partial E^{(e)}_{KL}}\right)_{T,E^{(P)}} = \left(\frac{\partial S_{KL}}{\partial T}\right)_{E^{(e)}} \ ,$$

and from equation (3.7.3) this may be written

$$\left(\frac{\partial S}{\partial E^{(e)}_{KL}}\right)_{T,E^{(P)}} = \psi_{KL}(T,E^{(e)}_{MN}) \ , \qquad (3.7.10)$$

where $\psi_{KL}$ is a tensor valued function of the temperature and the elastic strain.

A similar analysis using the form (3.5.4) of the entropy inequality gives

$$T = \left(\frac{\partial U}{\partial S}\right)_{E^{(e)},E^{(P)}} .$$

If the further assumption is made that the specific heat at constant strain [61]

$$C_E = \left(\frac{\partial U}{\partial T}\right)_{E^{(e)},E^{(P)}} = T\left(\frac{\partial S}{\partial T}\right)_{E^{(e)},E^{(P)}}$$

is independent of previous plastic straining then

$$\left(\frac{\partial S}{\partial T}\right)_{E^{(e)},E^{(P)}} = \phi(T,E_{KL}^{(e)}) , \tag{3.7.11}$$

where $\phi$ is a scalar function of the temperature and the elastic strain. Equations (3.7.10) and (3.7.11) show that

$$S = S^{(e)}(T,E_{RS}^{(e)}) + S^{(P)}(E_{RS}^{(P)}) \tag{3.7.12}$$

where $S^{(e)}$ and $S^{(P)}$ are the elastic and plastic parts of the entropy respectively.

From equations (3.7.8) and (3.7.9) the free energy may be written as

$$A = A^{(e)}(T,E_{KL}^{(e)}) + A^{(P)}(T,E_{KL}^{(P)}) \tag{3.7.13}$$

where $A^{(e)}$ is the elastic free energy and

$$A^{(P)} = - TS^{(P)} + A^{(P)}(E_{KL}^{(P)})$$

is the plastic free energy and $A^{(P)}$ is a function of the plastic strain and possibly the plastic strain history. From equations (3.7.8), (3.7.9), (3.7.12), and (3.7.13)

$$S^{(e)} = - \frac{\partial A^{(e)}}{\partial T} \tag{3.7.14}$$

and

$$S_{KL} = \rho_1 \frac{\partial A^{(e)}}{\partial E_{KL}^{(e)}} \quad . \tag{3.7.15}$$

The equation

$$\dot{A}^{(e)} = \frac{\partial A^{(e)}}{\partial E_{KL}^{(e)}} \dot{E}_{KL}^{(e)} + \frac{\partial A^{(e)}}{\partial T} \dot{T}$$

is valid whether or not plastic deformation is occurring and from equations (3.7.14) and (3.7.15) it becomes

$$\dot{A}^{(e)} = \frac{1}{\rho_1} S_{KL} \dot{E}_{KL}^{(e)} - S^{(e)} \dot{T} \quad .$$

For isothermal deformation

$$\dot{A}^{(e)} = \rho_1 \dot{W}^{(e)} \; ,$$

where $\dot{W}^{(e)} = S_{KL}\dot{E}^{(e)}_{KL}$ is the rate of increase of elastic strain energy per unit volume in C.1. If the material is elastically isotropic $S_{KL}$ and $E^{(e)}_{KL}$ are coaxial as are $S_{\alpha\beta}$ and $E^{(e)}_{\alpha\beta}$. Thus from equation (2.3.8) the result

$$S_{KL}\dot{E}^{(e)}_{KL} = S_{\alpha\beta}\hat{\dot{E}}^{(e)}_{\alpha\beta} = S_{\alpha\beta}\dot{E}^{(e)}_{\alpha\beta}$$

is obtained for an elastically isotropic material so that

$$\dot{W}^{(e)} = S_{\alpha\beta}\hat{\dot{E}}^{(e)}_{\alpha\beta} = \frac{\rho_1}{\rho_3}\,\sigma_{ij}d^{(e)}_{ij} \ . \tag{3.7.16}$$

The total rate of stress work per unit volume in C.1 is

$$\dot{W} = \frac{\rho_1}{\rho_3}\,\sigma_{ij}d_{ij} \ ,$$

so that from equation (2.3.10)

$$\dot{W} = \dot{W}^{(e)} + \dot{W}^{(P)},$$

where
$$\dot{W}^{(P)} = \frac{\rho_1}{\rho_3}\,\sigma_{ij}d^{*}_{ij} \tag{3.7.17}$$

is the rate of plastic energy dissipation per unit volume in C.1. If the material is elastically anisotropic then the rate of increase of elastic strain energy and the rate of plastic energy dissipation are

not given by equations (3.7.16) and (3.7.17).

Elastically isotropic materials only are considered in the remainder of this thesis.

# CHAPTER IV

# YIELD CONDITIONS AND FLOW RULES FOR PLASTIC DEFORMATION WITH FINITE ELASTIC STRAINS

## 4.1 Introduction

A flow rule is developed in section 4.2 which is based on the assumption that the materials to be considered satisfy Drucker's postulate [30]. The further assumptions of the existence of a yield function, no work-hardening, incompressibility,* elastic isotropy, and isothermal deformation are made. In section 4.3 the theory is limited to the discussion of materials whose elastic behavior is neo-Hookean and two yield conditions which are related to the classical von Mises yield condition are discussed. In section (4.4) a simplification which results from the further assumption of plastic isotropy is indicated.

## 4.2 A Plastic Flow Rule

In the classical elastic-plastic theory for perfectly plastic solids it is assumed that there exists a yield condition

$$f(\sigma_{ij},T) = 0 \qquad (4.2.1)$$

with the elastic domain defined by $f(\sigma_{ij},T) < 0$ .

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*The theory may be generalized to include finite elastic volume change [76].

If a material is plastically isotropic the form of the function f at a point must be independent of any rigid body rotation of the body and f is then a function of the invariants of the stress tensor.

However for a plastically anisotropic material there are two situations for which the form of the yield function $f(\sigma_{ij},T)$ is not satisfactory. If the elastic strains just prior to yield are small as in classical elasticity but the rotations are large, as in the problem of the elastica [6], or if the material undergoes finite elastic strains and rotations, the form of the yield function at a point in a plastically anisotropic material must depend on the rotation of the material element at that point. The yield condition is then of the form

$$f(\sigma_{KL},T) = 0 \tag{4.2.2}$$

proposed by Green and Naghdi [23]. The stress $\sigma_{KL}$ is given by

$$\sigma_{KL} = R_{iK}R_{jL}\sigma_{ij} ,$$

where $R_{iK}$ is a proper orthogonal tensor representing a pure rotation and is obtained from the polar decomposition of the total deformation gradient. That is

$$F_{iL} = R_{iK}U_{KL} ,$$

where $U_{KL}$ is a symmetric positive definite tensor representing a pure stretch deformation. Thus $\sigma_{KL}$ are the components of the stress tensor at a point referred to a Cartesian coordinate system which has undergone the same rotation as the material element at that point.

If large plastic deformations occur the yield condition for a plastically anisotropic material is of the form (4.2.2) regardless of the magnitude of the elastic strains.

The deformation rate tensors $d_{ij}$, $d_{ij}^{(e)}$, and $d_{ij}^{*}$ may also be referred to coordinates which have undergone the same rotations as the material elements so that

$$[d_{KL}, d_{KL}^{(e)}, d_{KL}^{*}] = R_{iK}R_{jL}[d_{ij}, d_{ij}^{(e)}, d_{ij}^{*}] ,$$

and since $R_{iK}$ is an orthogonal tensor it follows that

$$\sigma_{KL}[d_{KL}, d_{KL}^{(e)}, d_{KL}^{*}] = \sigma_{ij}[d_{ij}, d_{ij}^{(e)}, d_{ij}^{*}] .$$

Consider a body which is both elastically and plastically incompressible and elastically isotropic. Assume that a yield function $f(\sigma_{KL}, T)$ exists and suppose that at time $t^{o}$ the body is in equilibrium at the temperature T and homogeneous stress $\sigma_{KL}^{o}$ which lies inside the yield surface but sufficiently close to the yield surface that the addition of the stress necessary to move the stress point to the yield surface results only in infinitesimal additional elastic strains. Let

an external agency apply isothermally an additional stress to the body so as to move the stress point, as shown in Figure 4.1, to $\sigma^*_{KL}$ at time $t^*$, and then along the yield surface to the point $\sigma''_{KL}$ at time $t'' = t^* + \delta t$, and finally back to the original stress $\sigma^o_{KL}$ at time t. In the special loading case considered here the restriction is made

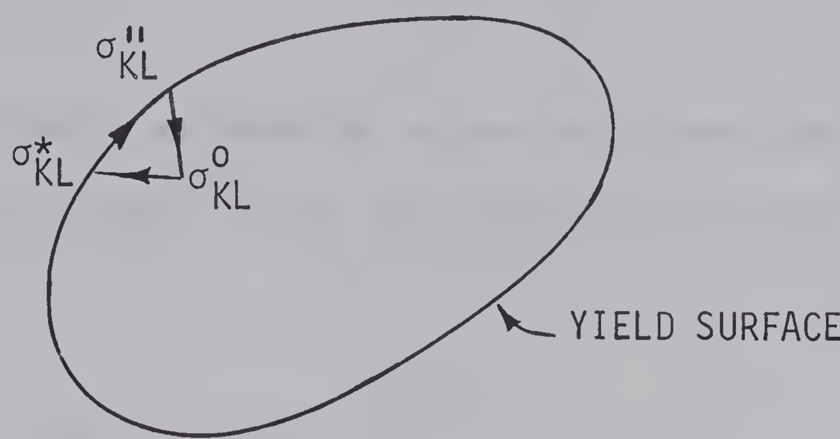



Figure 4.1

A Diagrammatic Representation of the

Isotropic Yield Surface in Stress Space

that the plastic strains occurring during the time interval $[t^*,t'']$ are infinitesimal and since the elastic strains are similarly restricted, changes in the configuration of the body during the cycle may be neglected.

According to Drucker's postulate the work done by the external agency during any complete cycle of loading and unloading is non-negative. The total work done per unit volume by both the original stress and the external agency is given by

$$W = \int_{t^0}^{t} \sigma_{KL} d_{KL} d\tau$$

$$= \int_{t^0}^{t} \sigma_{KL} (d_{KL}^{(e)} + d_{KL}^{*}) \, d\tau .$$

Since the plastic deformation occurs only from time $t^*$ to $t^*+\delta t$, and $d^*_{KL} \equiv 0$ at all other times, the expression for the total work done is

$$W = \oint \sigma_{KL} d_{KL}^{(e)} d\tau + \int_{t^*}^{t^*+\delta t} \sigma_{KL} d^*_{KL} d\tau ,$$

where $\oint$ denotes integration around the complete cycle of loading and unloading.

Since the material is elastically isotropic and incompressible it follows from the discussion in section 3.7 that

$$\oint \sigma_{KL} d_{KL}^{(e)} d\tau = \rho_3 \oint \, dA^{(e)} \equiv 0 ,$$

and thus

$$W = \int_{t^*}^{t^*+\delta t} \sigma_{KL} d^*_{KL} d\tau .$$

Furthermore since the changes in configuration of the body may be neglected it follows that the work done by the initial stress $\sigma^{o}_{KL}$ is given

by

$$W^{o} = \int_{t^*}^{t^*+\delta t} \sigma^{o}_{KL} d_{KL} d\tau \ .$$

Thus the net work done by the external agency during the complete cycle is

$$W - W^{o} = \int_{t^*}^{t^*+\delta t} (\sigma_{KL} - \sigma^{o}_{KL}) d^{*}_{KL} d\tau \ .$$

From Drucker's postulate

$$W - W^{o} \geq 0$$

so that

$$\lim_{\delta t \to 0} \frac{W - W^{o}}{\delta t} \geq 0 \ ,$$

consequently

$$(\sigma^{*}_{KL} - \sigma^{o}_{KL}) \ d^{*}_{KL} \geq 0 \ , \qquad (4.2.3)$$

where $\sigma^{*}_{KL}$ is a stress point on the yield surface associated with the plastic rate of deformation tensor $d^{*}_{KL}$ and $\sigma^{o}_{KL}$ is any stress point near $\sigma^{*}_{KL}$, on or just inside the yield surface. Equation (4.2.3) implies that the vector representing $d^{*}_{KL}$ in nine dimensional stress space is normal to the yield surface so that

$$d^{*}_{KL} = \lambda \frac{\partial f}{\partial \sigma_{KL}} , \tag{4.2.4}$$

where $\lambda$ is a non-negative factor of proportionality. For a plastically isotropic material this flow rule may be written as

$$d^{*}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} . \tag{4.2.5}$$

Equations (4.2.4) and (4.2.5) and the assumption of plastic incompressibility imply that the yield function must be independent of the hydrostatic pressure.

## 4.3 Two Yield Conditions and Associated Flow Rules

The remainder of this thesis is restricted to the discussion of elastic perfectly-plastic materials whose elastic behavior is neo-Hookean. That is, there exits an elastic potential given by

$$W = C(I_1^{(e)} - 3) , \tag{4.3.1}$$

where the invariant $I_1^{(e)}$, analogous to $I_1$ defined in section 3.2, is the first principal invariant of the tensor

$$C^{(e)}_{\alpha\beta} = 2E^{(e)}_{\alpha\beta} + \delta_{\alpha\beta}$$

so that

$$W = 2CE^{(e)}_{\alpha\alpha} \; .$$

The von Mises yield condition is

$$\frac{1}{2}\sigma'_{ij}\sigma'_{ij} - k^2 = 0 \; , \tag{4.3.2}$$

where k is the yield stress in pure shear. In the classical small strain elastic-plastic theory the condition (4.3.2) is equivalent to the condition that yielding occurs when the elastic shear strain energy reaches a critical value. The maximum shear strain energy yield condition is not equivalent to (4.3.2) for a material which may undergo finite elastic strains and the two yield conditions are henceforth referred to as the maximum shear strain energy yield condition and the von Mises yield condition respectively.

The flow rule for a plastically isotropic material which obeys the von Mises yield condition is, from equations (4.2.5) and (4.3.2),

$$d^*_{ij} = \lambda\sigma'_{ij} \; . \tag{4.3.3}$$

For the maximum shear strain energy yield condition yielding occurs when the elastic strain energy (equal to the shear strain energy since elastically incompressible materials only are considered) reaches a critical value W*.

From equation (3.4.5) for a neo-Hookean material

$$\sigma'_{ij} = 2C[F^{(e)}_{i\alpha}F^{(e)}_{j\alpha} - \frac{1}{3} I^{(e)}_1 \delta_{ij}] \,. \tag{4.3.4}$$

Using this constitutive equation and equation (4.3.1), the elastic shear strain energy may be expressed as a function of the invariants of the stress deviator tensor. Define two new stress invariants*

$$K_2 = \frac{1}{8C^2}\,\sigma'_{ij}\,\sigma'_{ij} = \frac{1}{4C^2}\,J'_2$$

and
$$K_3 = \frac{1}{24C^3}\,\sigma'_{ij}\sigma'_{jk}\sigma'_{ki} = \frac{1}{8C^3}\,J'_3 \,.$$

It may be shown that

$$K_2 = \frac{1}{3}\, I^{(e)^2}_1 - I^{(e)}_2 \tag{4.3.5}$$

and
$$K_3 = 1 - \frac{1}{3}\, I^{(e)}_1 I^{(e)}_2 + \frac{2}{27}\, I^{(e)^3}_1 \,. \tag{4.3.6}$$

Using equation (4.3.5) to eliminate $I^{(e)}_2$ from equation (4.3.6) and rearranging gives

$$K_2 I^{(e)}_1 - \frac{1}{9}\, I^{(e)^3}_1 + 3(1-K_3) = 0$$

---

*$K_2$ and $K_3$ are called stress invariants even though they are dimensionless, since they are derived from the stress deviator invariants $J'_2$ and $J'_3$ .

or $$K_2(\frac{W}{C} + 3) - \frac{1}{9}(\frac{W}{C} + 3)^3 + 3(1-K_3) = 0 \ .$$

Consequently the maximum shear strain energy yield condition for a neo-Hookean material is

$$f = K_2(\frac{W^*}{C} + 3) - \frac{1}{9}(\frac{W^*}{C} + 3)^3 + 3(1-K_3) = 0 \qquad (4.3.7)$$

and it may be verified that $f < 0$ when $W < W^*$.

Also if yielding occurs after infinitesimal elastic strains then from equation (4.3.4)

$$\sigma'_{ij} \simeq 4C \, e'_{ij} \ ,$$

$$W \simeq \frac{1}{8C} \sigma'_{ij}\sigma'_{ij} \ ,$$

and $W^* << C$. Therefore the yield condition (4.3.7) becomes

$$3K_2 - 3\,\frac{W^*}{C} + 0(\frac{W^{*2}}{C^2}) = 0$$

so that if terms $0(\frac{W^{*2}}{C^2})$ are neglected this yield condition reduces to the von Mises yield condition.

The yield function f for the maximum shear strain energy yield condition is a function of the two stress invariants $K_2$ and $K_3$, and since the material is incompressible, the associated flow rule

(4.2.5) for this yield condition is

$$d^*_{ij} = \lambda\left[\frac{\partial f}{\partial K_2}\frac{\sigma'_{ij}}{4C^2} + \frac{\partial f}{\partial K_3}\left(\frac{1}{8C^3}\sigma'_{ik}\sigma'_{kj} - \frac{K_2}{3C}\delta_{ij}\right)\right]$$

$$= \lambda\left[\left(\frac{W^*}{C} + 3\right)\frac{\sigma'_{ij}}{4C^2} - \left(\frac{3\sigma'_{ik}\sigma'_{kj}}{8C^3} - \frac{K_2}{C}\delta_{ij}\right)\right] . \qquad (4.3.8)$$

## 4.4 A Further Simplification Due to Isotropy

Using equation (2.3.11) the flow rule for a plastically isotropic material is

$$\frac{1}{2}\left[F^{(e)}_{i\alpha} d^{(P)}_{\alpha\beta} F^{(e)^{-1}}_{\beta j} + F^{(e)}_{j\beta} d^{(P)}_{\alpha\beta} F^{(e)^{-1}}_{\alpha i}\right] = \lambda \frac{\partial f}{\partial \sigma'_{ij}} .$$

A significant simplification of this flow rule results if the elastic deformation gradient tensor is expressed in the polar form

$$F^{(e)}_{i\alpha} = V^{(e)}_{ij} R^{(e)}_{j\alpha} ,$$

where $R^{(e)}_{j\alpha}$ is the previously defined proper orthogonal tensor representing the rigid body rotation of C.3 relative to C.2 and $V^{(e)}_{ij}$ is a symmetric positive definite tensor such that

$$V^{(e)}_{ik} V^{(e)}_{kj} = F^{(e)}_{i\alpha} F^{(e)}_{j\alpha} . \qquad (4.4.1)$$

With the use of equation (4.4.1) the constitutive equation

(4.3.4) becomes

$$\sigma'_{ij} = 2C[V^{(e)}_{ik}V^{(e)}_{kj} - \frac{1}{3}(V^{(e)}_{mk}V^{(e)}_{mk})\delta_{ij}] \tag{4.4.2}$$

and the expression for $d^*_{ij}$ becomes

$$d^*_{ij} = \frac{1}{2}[V^{(e)}_{ik}d^{(P)}_{km}V^{(e)^{-1}}_{mj} + V^{(e)}_{jk}d^{(P)}_{km}V^{(e)^{-1}}_{mi}] , \tag{4.4.3}$$

where $$d^{(P)}_{km} = R^{(e)}_{k\alpha}R^{(e)}_{m\beta}d^{(P)}_{\alpha\beta} . \tag{4.4.4}$$

From the constitutive equation (4.4.2) the tensors $\sigma'_{ij}$ and $V^{(e)}_{ij}$ are coaxial and since the materials considered are plastically isotropic, f is a function of the stress invariants and the tensors $\sigma'_{ij}$ and $d^*_{ij}$ are also coaxial. It may thus be deduced from equation (4.4.3) (see Appendix A) that the tensors $d^*_{ij}$ and $d^{(P)}_{ij}$ are also coaxial so that equation (4.4.3) reduces to

$$d^*_{ij} = d^{(P)}_{ij} \tag{4.4.5}$$

for a plastically isotropic material.

Furthermore since

$$F_{iK} = V^{(e)}_{ij}R^{(e)}_{j\alpha}F^{(P)}_{\alpha K}$$

and $$d^{(P)}_{\alpha\beta} = \frac{1}{2}(\dot{F}^{(P)}_{\alpha K}F^{(P)^{-1}}_{K\beta} + \dot{F}^{(P)}_{\beta K}F^{(P)^{-1}}_{K\alpha}) ,$$

it follows from equation (4.4.3) and (4.4.4) that

$$d_{ij}^{(P)} = \frac{1}{2}\,(-V_{ik}^{(e)^{-1}}\dot{V}_{kj}^{(e)} - V_{jk}^{(e)^{-1}}\dot{V}_{ki}^{(e)} + V_{ik}^{(e)^{-1}}\dot{F}_{kL}F_{Lm}^{-1}V_{mj}^{(e)}$$

$$+ V_{jk}^{(e)^{-1}}\dot{F}_{kL}F_{Lm}^{-1}V_{mi}^{(e)}) \ . \qquad (4.4.6)$$

# CHAPTER V

# SIMPLE SHEAR PROBLEM

## 5.1 Introduction

Simple shear of a cuboid is considered in this chapter. The sides in C.1 are parallel to the axes of a fixed Cartesian coordinate system. After deformation the position of a particle originally at $X_K$ is given by

$$x_1 = X_1 + KX_2 ,$$

$$x_2 = X_2 ,$$

and $$x_3 = X_3 .$$

This is a homogeneous isochoric deformation and the parameter $K = \tan\theta$, where $\theta$ is the angle of shear as shown in Figure 5.1, is used as a time scale. The deformation is assumed to occur isothermally.

The total deformation gradient and its total derivative with respect to the time like parameter K are

$$[F_{iK}] = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } [\dot{F}_{iK}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad (5.1.1a,b)$$

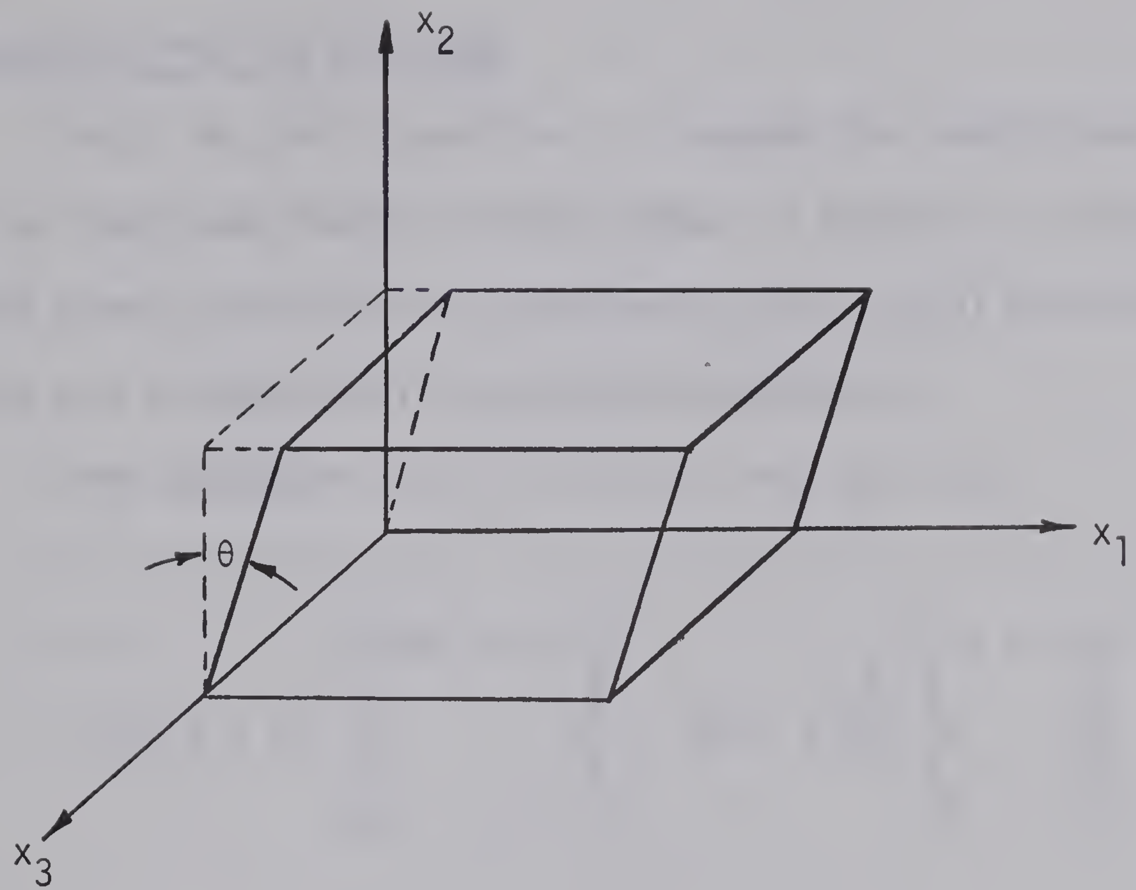



Figure 5.1

Simple Shear of a Cuboid

The deformation is completely specified by the parameter K.

The material is assumed to be elastic-perfectly plastic and to exhibit neo-Hookean behavior for loading up to the yield point and for unloading from an elastic-plastic state. Both the von Mises and the maximum shear strain energy yield conditions are considered.

The elastic-plastic simple shear problem consists of finding the stress deviator tensor and the elastic stretch tensor $V_{ij}^{(e)}$ as functions of the parameter K. Complete solutions are found numerically for both yield conditions.

## 5.2 Solution Prior to Yielding

Until the yield condition is reached the cuboid remains elastic so that upon removal of the stress it returns to configuration C.1. The elastic solution was obtained by Rivlin [63] for both compressible and incompressible hyperelastic materials.

From equations (4.4.1), (4.4.2), and (5.1.1a)

$$[\sigma'_{ij}] = 2C \begin{bmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2C(1 + \frac{K^2}{3}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\sigma'_{11} = \frac{4C}{3} K^2 \ ,$$

$$\sigma'_{22} = \sigma'_{33} = -\frac{2C}{3} K^2 \ ,$$

$$\sigma'_{12} = 2CK \ , \qquad (5.2.1a,b,c)$$

and all other stress deviator components are zero.

If the stress on one of the faces of the cuboid is specified then the components of the stress tensor may be determined from equations (5.2.1a,b,c).

Consider first the von Mises yield condition (4.3.2) which using equations (5.2.1a,b,c) becomes

$$K^4 + 3K^2 - \frac{3}{4}\left(\frac{k}{C}\right)^2 = 0 \ .$$

Solving for the value of K at yield gives

$$K_y = \left(-\frac{3}{2} + \frac{1}{2}\sqrt{9 + 3\left(\frac{k}{C}\right)^2}\,\right)^{1/2} \ . \tag{5.2.2}$$

For the maximum shear strain energy yield condition $K_y$ must be determined as a function of $\frac{W^*}{C}$ . From equation (5.1.1a) it follows that

$$I_1^{(e)} = 3 + K^2 \ ,$$

consequently at yield

$$\frac{W^*}{C} + 3 = 3 + K_y^{\ 2}$$

or

$$K_y = \sqrt{\frac{W^*}{C}} \ . \tag{5.2.3}$$

A basis for comparison is needed for the solutions with the two yield conditions and the yield stress in simple tension is chosen arbitrarily to be this basis.

For this purpose consider the uniform extension [64] of a block of an incompressible neo-Hookean material due to a force parallel to the $x_1$ direction. The deformation is specified by

$$x_1 = \Lambda X_1$$

$$x_2 = \frac{1}{\sqrt{\Lambda}} X_2 \ ,$$

and
$$x_3 = \frac{1}{\sqrt{\Lambda}} X_3 \ ,$$

so that the deformation gradient has components

$$[F_{iK}] = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 1/\sqrt{\Lambda} & 0 \\ 0 & 0 & 1/\sqrt{\Lambda} \end{bmatrix} . \tag{5.2.4}$$

It follows from equation (4.3.4) that

$$[\sigma_{ij}] = \begin{bmatrix} 2C\Lambda^2+p & 0 & 0 \\ 0 & \frac{2C}{\Lambda}+p & 0 \\ 0 & 0 & \frac{2C}{\Lambda}+p \end{bmatrix} .$$

If the faces parallel to the direction of extension are stress free then

$$p = -\frac{2C}{\Lambda}$$

and
$$\sigma_{11} = 2C(\Lambda^2 - \frac{1}{\Lambda}) . \tag{5.2.5}$$

The yield stress Y in simple tension for a von Mises material is given by

$$Y = \sqrt{3}\, k \; . \tag{5.2.6}$$

Consequently if the material is neo-Hookean up to yielding, it will yield in simple tension when the extension is the solution $\Lambda_y$ of the equation

$$\Lambda_y^3 - \frac{\sqrt{3}}{2} \left(\frac{k}{C}\right) \Lambda_y - 1 = 0 \; , \tag{5.2.7}$$

which results from equations (5.2.5) and (5.2.6).

Using equation (5.2.4) the invariant $I_1^{(e)}$ is found to be

$$I_1^{(e)} = \Lambda^2 + \frac{2}{\Lambda} \tag{5.2.8}$$

for uniform extension. Thus if a material whose yielding is governed by the maximum shear strain energy yield condition is to yield in simple tension at the same stress and therefore at the extension $\Lambda_y$ found from equation (5.2.7), the material constant $(\frac{W^*}{C})$ must satisfy the equation

$$\frac{W^*}{C} = \Lambda_y{}^2 + \frac{2}{\Lambda_y} - 3 \; . \tag{5.2.9}$$

In the remainder of this chapter and in Chapter VI, all the stresses are non-dimensionalized by division by the shear yield stress k. The values of k used for the materials obeying the maximum strain energy yield condition are those determined from equations (5.2.7) and (5.2.9) for von Mises materials which have the same yield stress in simple tension.

## 5.3 Material Obeying the von Mises Yield Condition

The elastic-perfectly plastic solution is first obtained for a material which is neo-Hookean up to the yield point and during unloading from the yield point, which is determined by the von Mises yield condition.

Using equations (4.4.2) and (4.4.6) the flow rule (4.3.3) becomes

$$\underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{V}}^{(e)} + \left(\underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{V}}^{(e)}\right)^T = \underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1}\underset{\sim}{V}^{(e)} + \left(\underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1}\underset{\sim}{V}^{(e)}\right)^T$$

$$- \frac{4\lambda}{(k/C)}\left[\underset{\sim}{V}^{(e)^2} - \frac{1}{3}\,\mathrm{tr}\left(\underset{\sim}{V}^{(e)^2}\right)\underset{\sim}{I}\right] , \qquad (5.3.1)$$

which is written in matrix notation with the superscript T denoting the transpose, $\underset{\sim}{I}$ the identity matrix, and tr(.) the trace of the argument matrix. The non-negative parameter $\lambda$ in equation (5.3.1) differs from that in equation (4.3.3) by a factor k.

The von Mises yield condition after substitution of equation (4.4.2) becomes

$$\mathrm{tr}[\frac{4}{(k/C)^2}\{\underset{\sim}{V}^{(e)^4} - \frac{2}{3}\,\mathrm{tr}(\underset{\sim}{V}^{(e)^2})\,\underset{\sim}{V}^{(e)^2} + \frac{1}{9}\,\mathrm{tr}^2(\underset{\sim}{V}^{(e)^2})\underset{\sim}{I}\}] - 2 = 0$$

and taking the total derivative with respect to the time like parameter K gives

$$\frac{D}{DK}\,[\mathrm{tr}(\underset{\sim}{V}^{(e)^4}) - \frac{1}{3}\,\mathrm{tr}^2(\underset{\sim}{V}^{(e)^2})] = 0$$

or

$$\mathrm{tr}(\underset{\sim}{V}^{(e)^3}\dot{\underset{\sim}{V}}^{(e)}) - \frac{1}{3}\,\mathrm{tr}(\underset{\sim}{V}^{(e)^2})\mathrm{tr}(\underset{\sim}{V}^{(e)}\dot{\underset{\sim}{V}}^{(e)}) = 0\ . \qquad (5.3.2)$$

The elastic solution (5.2.1) is valid until K reaches $K_y$ as determined from equation (5.2.2). For $K = K_y$ the matrix $\underset{\sim}{V}^{(e)}$ and the components of the stress deviator are known. For $K > K_y$ the components of $\underset{\sim}{V}^{(e)}$ are determined from the system of non-linear ordinary differential equations (5.3.1) and (5.3.2) using the value of $\underset{\sim}{V}^{(e)}$ at $K = K_y$ as the initial condition. The numerical method used to obtain a solution is described in section (5.5).

## 5.4 Material Obeying the Maximum Shear Strain Energy Yield Condition

The solution for the simple shear of a cuboid of a neo-Hookean elastic-perfectly plastic material which obeys the maximum shear strain energy yield condition is obtained in a manner similar to that discussed in the previous section for the von Mises yield condition.

Substitution of equation (4.4.2) into the expressions (4.3.5)

and (4.3.6) for the stress invariants $K_2$ and $K_3$ gives

$$K_2 = \frac{1}{2}\,[\mathrm{tr}(\underset{\sim}{V}^{(e)^4}) - \frac{1}{3}\,\mathrm{tr}^2(\underset{\sim}{V}^{(e)^2})] \qquad (5.4.1)$$

and

$$K_3 = \frac{1}{3}\,[\mathrm{tr}(\underset{\sim}{V}^{(e)^6}) - \mathrm{tr}(\underset{\sim}{V}^{(e)^4})\mathrm{tr}(\underset{\sim}{V}^{(e)^2})$$

$$+ \frac{2}{9}\,\mathrm{tr}^2\,(\underset{\sim}{V}^{(e)^2})] \,. \qquad (5.4.2)$$

Using equations (4.4.2), (4.4.5), and (4.4.6) the flow rule (4.3.8) becomes

$$\underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{V}}^{(e)} + (\underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{V}}^{(e)})^T = \underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1}\underset{\sim}{V}^{(e)} + (\underset{\sim}{V}^{(e)^{-1}}\dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1}\underset{\sim}{V}^{(e)})^T$$

$$- \lambda(\frac{k}{C})\{(\frac{W^*}{C} + 3)[\underset{\sim}{V}^{(e)^2} - \frac{1}{3}\,\mathrm{tr}(\underset{\sim}{V}^{(e)^2})\underset{\sim}{I}] - 3[\underset{\sim}{V}^{(e)^2} - \frac{1}{3}\,\mathrm{tr}(\underset{\sim}{V}^{(e)^2})\underset{\sim}{I}]^2$$

$$+ \mathrm{tr}([\underset{\sim}{V}^{(e)^2} - \frac{1}{3}\,\mathrm{tr}(\underset{\sim}{V}^{(e)^2})\underset{\sim}{I}]^2)\} \qquad (5.4.3)$$

and the maximum shear strain energy yield condition (4.3.7) is, after substitution of equations (5.4.1) and (5.4.2)

$$\frac{1}{2}\,(\frac{W^*}{C} + 3)[\mathrm{tr}(\underset{\sim}{V}^{(e)^4}) - \frac{1}{3}\,\mathrm{tr}^2(\underset{\sim}{V}^{(e)^2})] - \frac{1}{9}\,(\frac{W^*}{C} + 3)^3 + 3$$

$$- [\mathrm{tr}(\underset{\sim}{V}^{(e)^6}) - \mathrm{tr}(\underset{\sim}{V}^{(e)^4})\mathrm{tr}(\underset{\sim}{V}^{(e)^2}) + \frac{2}{9}\,\mathrm{tr}^3(\underset{\sim}{V}^{(e)^2})] = 0 \,.$$

Taking the total derivative with respect to K gives

$$\left(\frac{W^*}{C} + 3\right)\left[\mathrm{tr}(\underset{\sim}{V}^{(e)^3}\dot{\underset{\sim}{V}}^{(e)}) - \frac{1}{3}\,\mathrm{tr}(\underset{\sim}{V}^{(e)^2})\mathrm{tr}(\dot{\underset{\sim}{V}}^{(e)}\underset{\sim}{V}^{(e)})\right]$$

$$+ \left[\mathrm{tr}(\underset{\sim}{V}^{(e)^4}) - \frac{2}{3}\,\mathrm{tr}^2(\underset{\sim}{V}^{(e)^2})\right]\mathrm{tr}(\underset{\sim}{V}^{(e)}\dot{\underset{\sim}{V}}^{(e)})$$

$$- 3\mathrm{tr}(\underset{\sim}{V}^{(e)^5}\dot{\underset{\sim}{V}}^{(e)}) + 2\mathrm{tr}(\underset{\sim}{V}^{(e)^3}\dot{\underset{\sim}{V}}^{(e)})\mathrm{tr}(\underset{\sim}{V}^{(e)^2}) = 0\ . \qquad (5.4.4)$$

Equations (5.4.3) and (5.4.4) are solved numerically for $\underset{\sim}{V}^{(e)}$ as a function of K using the value of $\underset{\sim}{V}^{(e)}$ at yield as the initial condition.

## 5.5 Numerical Solution

The two systems of non-linear ordinary differential equations (5.3.1), (5.3.2) and (5.4.3), (5.4.4) are sufficiently complex that closed form solutions do not appear possible. Instead a numerical technique, which is described in this section, was used to obtain the solutions.

The following discussion is for the elastic-perfectly plastic solution for the von Mises material and the same method was used for the material with the maximum shear strain energy yield condition.

Consider the matrix equation (5.3.1). Since both sides are symmetric 3 x 3 matrices it represents six independent equations with seven unknowns; the six independent components of $\underset{\sim}{V}^{(e)}$ and the parameter $\lambda$. The yield condition provides the additional equation (5.3.2)

for the determination of $\lambda$.

The Runge-Kutta method [65], generalized for a system of simultaneous first order differential equations was used to solve equations (5.3.1) and (5.3.2). Consider a system of m differential equations.

$$\frac{d}{dt} y^{(1)} = g^{(1)} (t, y^{(1)}, y^{(2)}, \ldots, y^{(m)}) ,$$

$$\frac{d}{dt} y^{(2)} = g^{(2)} (t, y^{(1)}, y^{(2)}, \ldots, y^{(m)}) ,$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d}{dt} y^{(m)} = g^{(m)} (t, y^{(1)}, y^{(2)}, \ldots, y^{(m)}) . \tag{5.5.1}$$

For an integration step size $\Delta t$, the standard fourth order Runge-Kutta formula at the $n^{th}$ integration step is

$$y_{n+1}^{(i)} = y_n^{(i)} + \frac{\Delta t}{6} (h_1^{(i)} + 2h_2^{(i)} + 2h_3^{(i)} + h_4^{(i)}) + O(\Delta t^5)$$

where

$$h_1^{(i)} = g^{(i)} (t_n, y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)}) ,$$

$$h_2^{(i)} = g^{(i)}(t_n + \frac{\Delta t}{2}, y_n^{(1)} + \frac{\Delta t}{2} h_1^{(1)}, y_n^{(2)} + \frac{\Delta t}{2} h_1^{(2)}, \ldots, y_n^{(m)} + \frac{\Delta t}{2} h_1^{(m)}) ,$$

$$h_3^{(i)} = g^{(i)}(t_n + \frac{\Delta t}{2}, y_n^{(1)} + \frac{\Delta t}{2} h_2^{(1)}, y_n^{(2)} + \frac{\Delta t}{2} h_2^{(2)}, \ldots, y_n^{(m)} + \frac{\Delta t}{2} h_2^{(m)}) ,$$

and

$$h_4^{(i)} = g^{(i)} (t_n + \Delta t, y_n^{(1)} + \Delta t h_3^{(1)}, y_n^{(2)} + \Delta t h_3^{(2)}, \ldots, y_n^{(m)} + \Delta t h_3^{(m)}).$$

The derivation [65] of these formulae is lengthy and is not included here.

The system of differential equations (5.3.1) and (5.3.2) cannot be put in the form (5.5.1) since $\dot{\underset{\sim}{V}}^{(e)}$ cannot be written explicitly as a function of K and $\underset{\sim}{V}^{(e)}$. For given values of K and $\underset{\sim}{V}^{(e)}$ however, it is possible to determine the value of the matrix $\dot{\underset{\sim}{V}}^{(e)}$ from equations (5.3.1) and (5.3.2) by guessing a value for $\lambda$ and solving equation (5.3.1) as a system of algebraic equations for the components of $\dot{\underset{\sim}{V}}^{(e)}$. The resulting values of $\dot{\underset{\sim}{V}}^{(e)}$ and the given values of $\underset{\sim}{V}^{(e)}$ are then substituted into equation (5.3.2). If the latter equation is not satisfied, then the value of $\lambda$ is varied until the resulting matrix $\dot{\underset{\sim}{V}}^{(e)}$ does satisfy equation (5.3.2). It is thus possible to determine the value of the matrix $\dot{\underset{\sim}{V}}^{(e)}$ as a function of $V^{(e)}$ and K so that from a numerical point of view the system of equations (5.3.1) and (5.3.2) is equivalent to a system of the form

$$\dot{\underset{\sim}{V}}^{(e)} = \underset{\sim}{G}(K, \underset{\sim}{V}^{(e)})$$

even though the explicit form of the matrix function $\underset{\sim}{G}$ cannot be found.

Thus the fourth order Runge-Kutta formula for this problem, in matrix notation at the $n^{th}$ integration step is

$$\underset{\sim}{V}_{n+1}^{(e)} = \underset{\sim}{V}_{n}^{(e)} + \frac{\Delta K}{6} (\underset{\sim}{H}_1 + 2\underset{\sim}{H}_2 + 2\underset{\sim}{H}_3 + \underset{\sim}{H}_4), \qquad (5.5.2)$$

where
$$\underset{\sim}{H}_1 = \underset{\sim}{G}(K, \underset{\sim}{V}_{n}^{(e)})$$

$$\underset{\sim}{H}_2 = \underset{\sim}{G}(K + \frac{\Delta K}{2}, \underset{\sim}{V}_{n}^{(e)} + \frac{\Delta K}{2} \underset{\sim}{H}_1),$$

$$\underset{\sim}{H}_3 = \underset{\sim}{G}(K + \frac{\Delta K}{2}, \underset{\sim}{V}_{n}^{(e)} + \frac{\Delta K}{2} \underset{\sim}{H}_2),$$

$$\underset{\sim}{H}_4 = \underset{\sim}{G}(K + \Delta K, \underset{\sim}{V}_{n}^{(e)} + \Delta K \underset{\sim}{H}_3),$$

and $\Delta K$ is the integration stepsize.

A computer program was written employing the above method to solve numerically the systems of equations for both yield conditions. Solutions were found for various values of k/C and the corresponding values of $\frac{W^*}{C}$, with total integration intervals extending to values of K as high as two.

The results are shown graphically in Figures 5.3 to 5.8.

Estimates of the truncation error due to neglect of terms

$O(\Delta K^5)$ in equation (5.5.2) are available [66], but since they are for the most general case, that is a system such as (5.5.1), they tend to be pessimistic and rather difficult to apply to a complex system such as (5.3.1) and (5.3.2). An indication of the truncation error may be obtained however by changing the integration step size $\Delta K$.

All the simple shear solutions found were obtained using step sizes of 0.01 and 0.005. A comparison of the corresponding results shows complete agreement to four significant figures in the values of $\underset{\sim}{V}^{(e)}$ and the maximum difference observed in the values of the non-dimensionalized stress deviator is $10^{-4}$.

In an attempt to reduce the possibility of round off error affecting the results, double precision (16 significant figures) was used for all calculations and it is assumed that the effects of round off error are negligible.

## 5.6 Discussion

The results indicate that as k/C becomes smaller the normal stresses become smaller and are negligible for $k/C \ll 1$. Also for $k/C \ll 1$ the results approach those obtained from the classical elastic-perfectly plastic theory.

For the von Mises yield condition the stress deviator component $\sigma'_{33}$ approaches zero for values of K large compared with $K_y$ and if $\sigma_{33} = 0$ then $\sigma_{11} = -\sigma_{22} = \sigma'_{11}$ for large K. The maximum shear strain energy yield condition gives quite different normal stress effects during elastic-plastic flow. The relation $\sigma'_{22} = \sigma'_{33}$

which holds for elastic deformation prior to yielding also holds for elastic-plastic deformation for all K and if $\sigma_{33} = 0$ then $\sigma_{22} = 0$ and $\sigma_{11} = -3\sigma'_{33}$ .

One further result is of interest. Figure 5.2 which for simplicity considers only the $x_1$ and $x_2$ directions, shows a typical stress deviator system in C.3 during elastic-plastic flow for either the von Mises or maximum shear strain energy yield conditions. The figure shows the configuration C.2 which results from irrotational

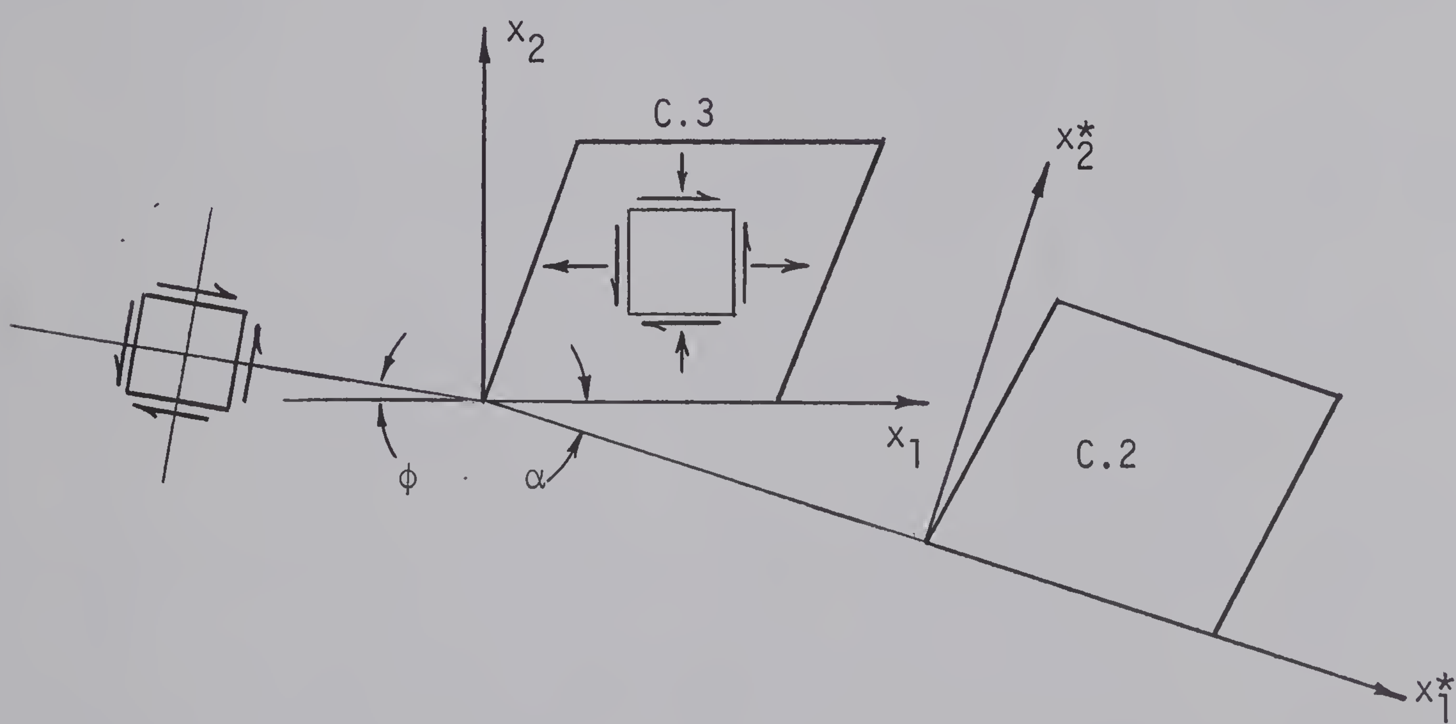
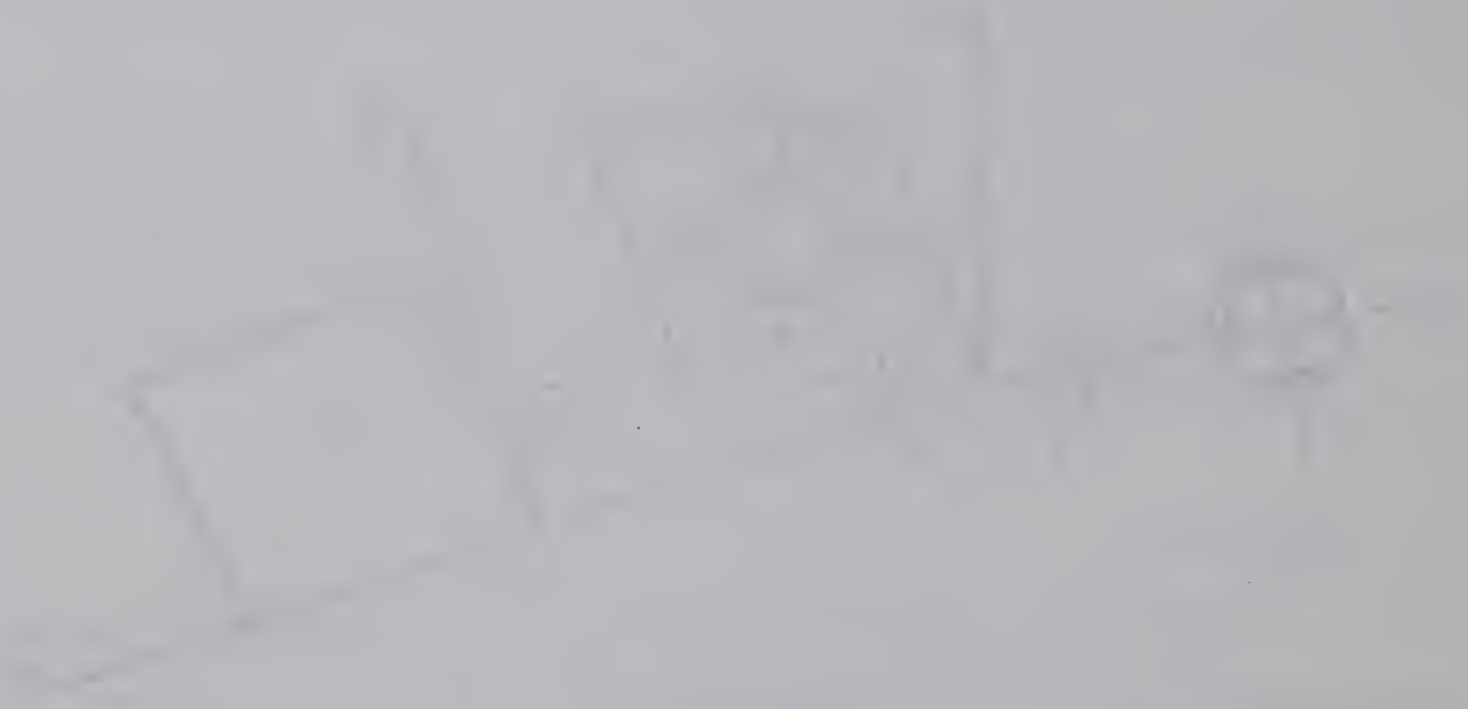



Figure 5.2

Direction of Maximum Shear Stress and the Orientation of C.2 Resulting from Irrotational Elastic Unloading

elastic unloading from C.3 with $\alpha$ being the angle through which lines of constant $X_2$ rotate during unloading from C.3 to C.2. The angle $\phi$ indicates the direction of maximum shearing stress in C.3. From the



graphs of $\frac{1}{k}\sigma'_{ij}$ versus K it is seen that $\phi$ approaches a constant value for large K since the stress deviator components approach constant values. It is found however that $\alpha$ approaches $\phi$ as K becomes large compared to $K_y$ so that C.2 tends to align itself so that the axes $ox_i^*$ are in the directions of the maximum shearing stress.

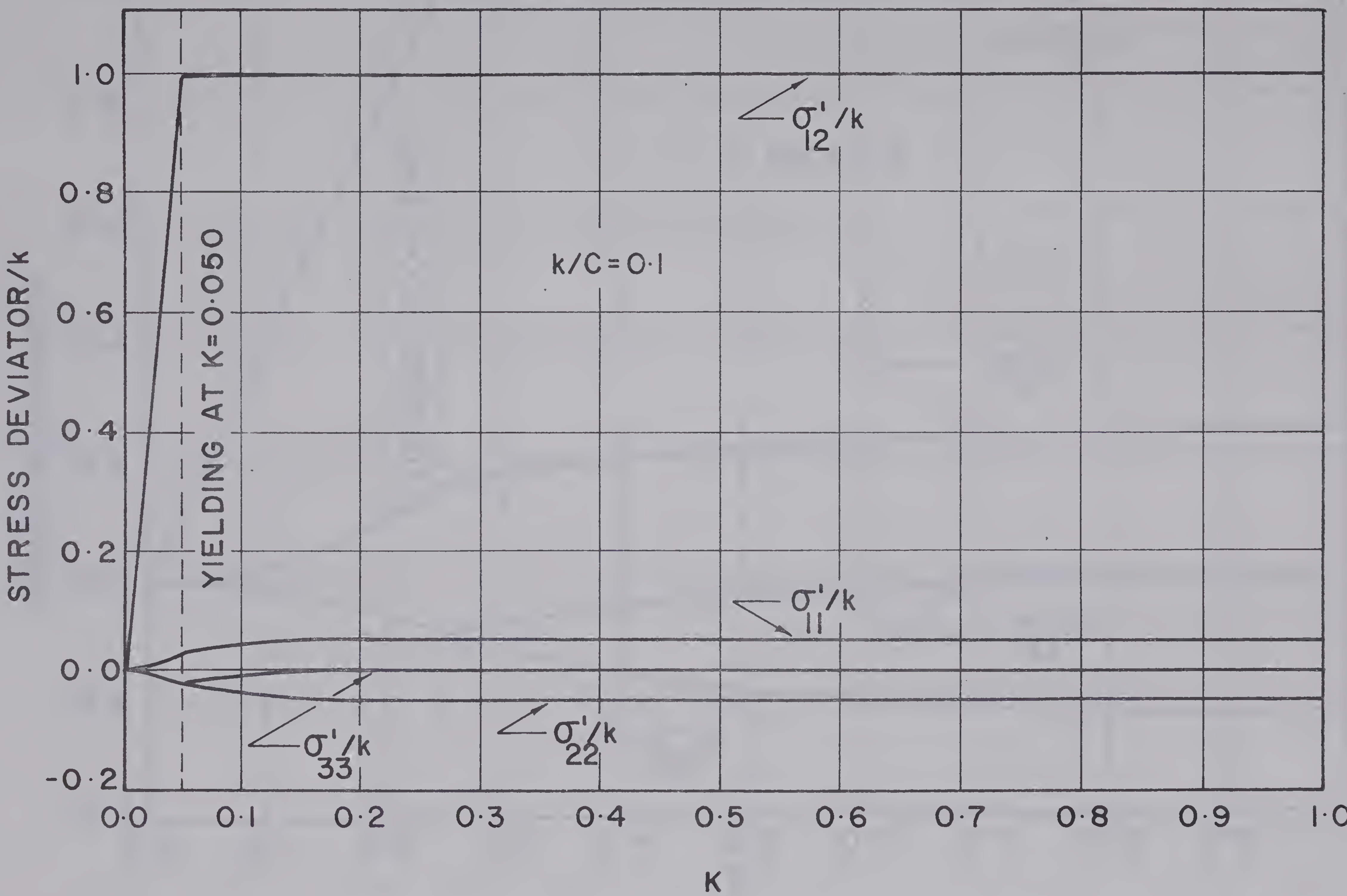



Figure 5.3 von Mises Yield Condition, $\sigma'_{ij}/k$ versus K

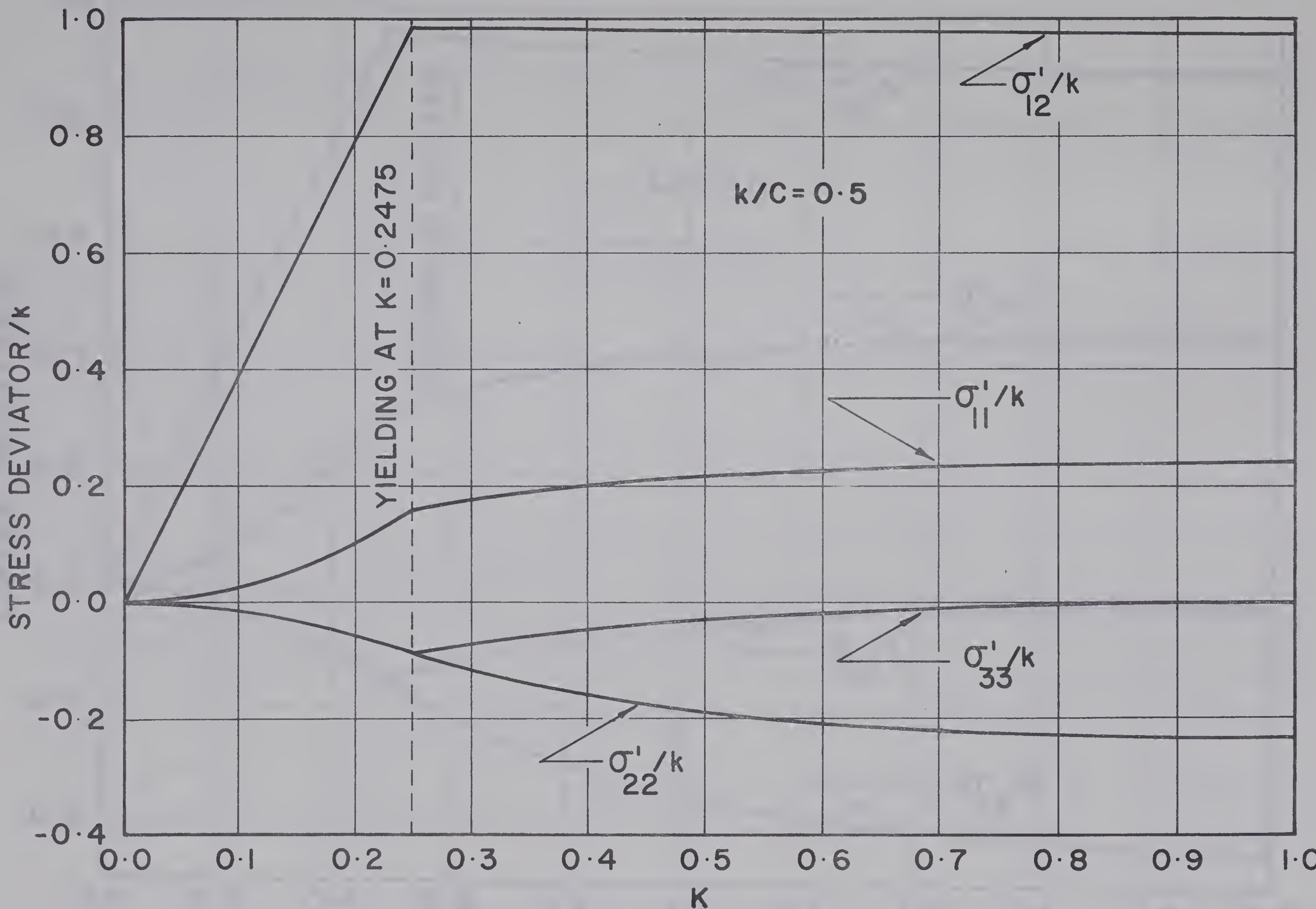



Figure 5.4 von Mises Yield Condition, $\sigma'_{ij}/k$ versus K

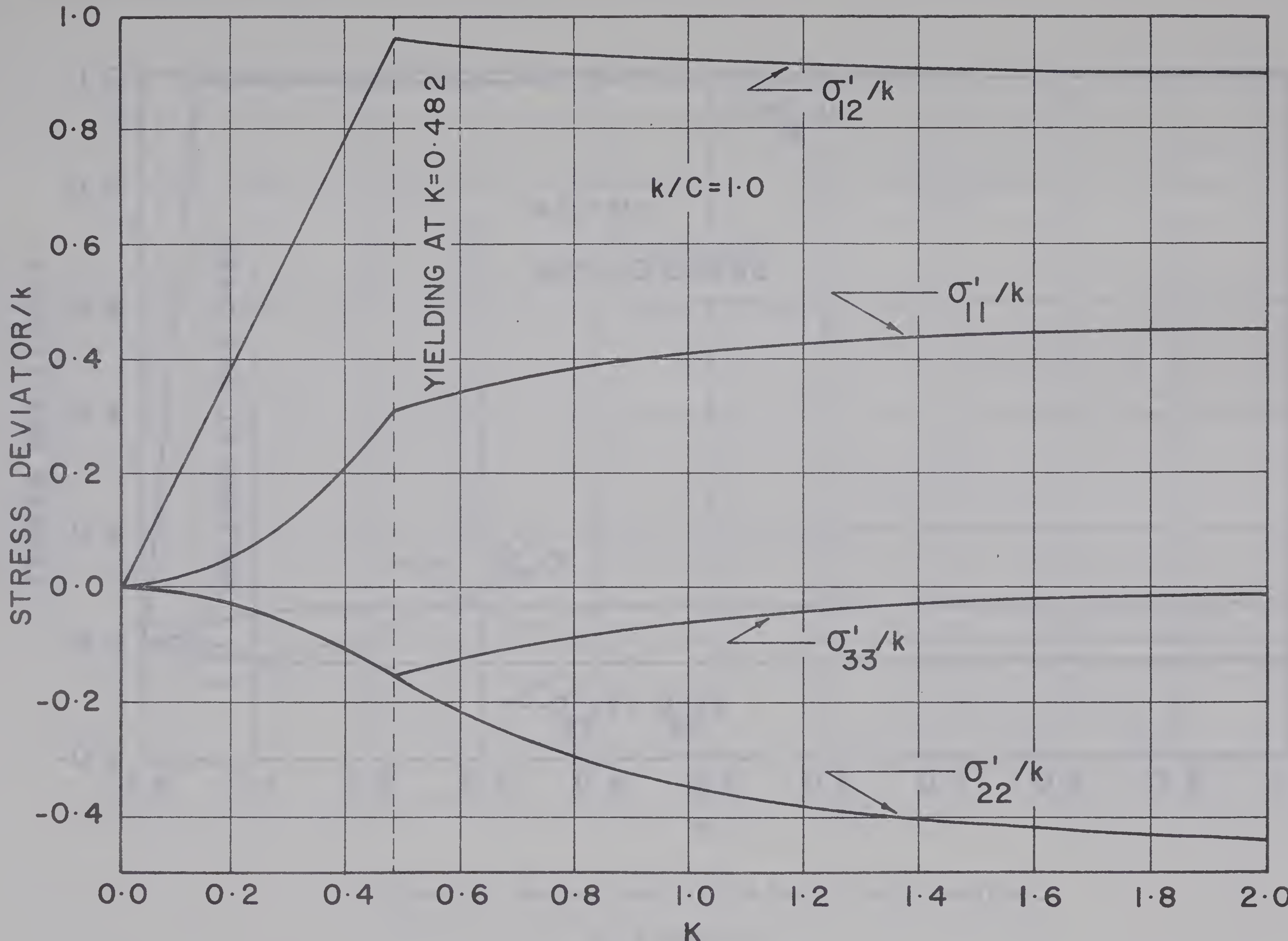
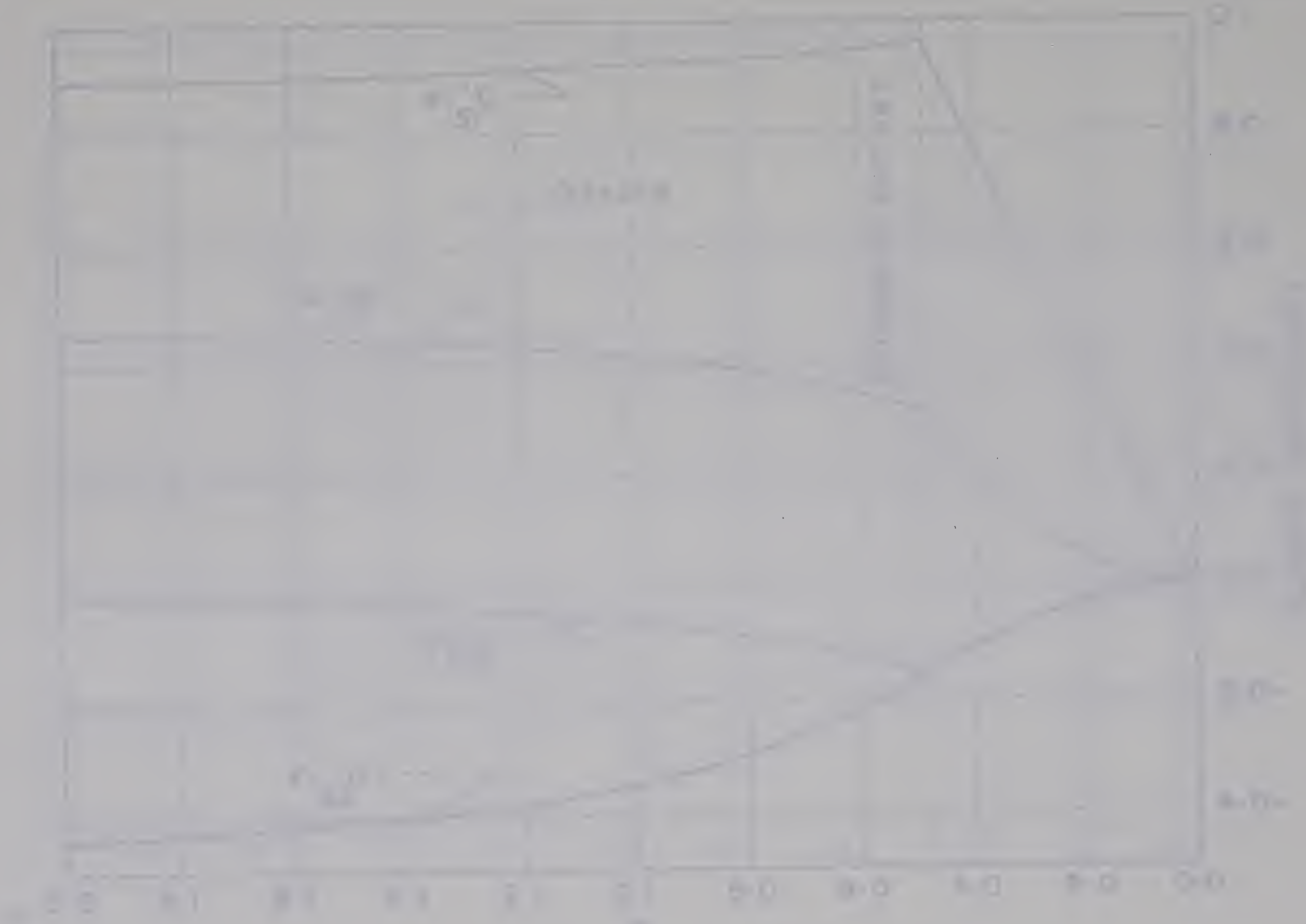



Figure 5.5 von Mises Yield Condition, $\sigma'_{ij}/k$ versus K



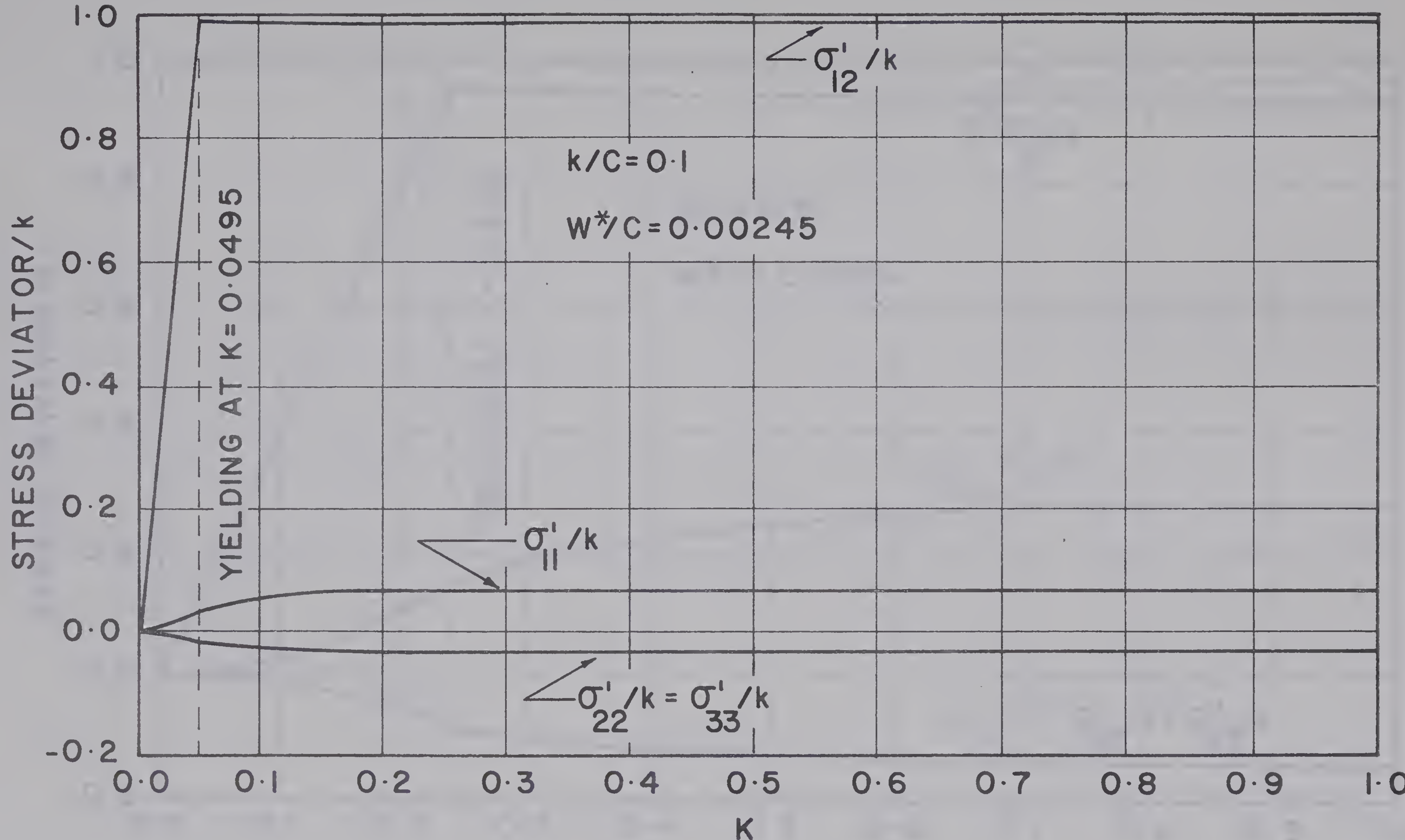



Figure 5.6 Maximum Shear Strain Energy Yield Condition,

$\sigma'_{ij}/k$ versus K

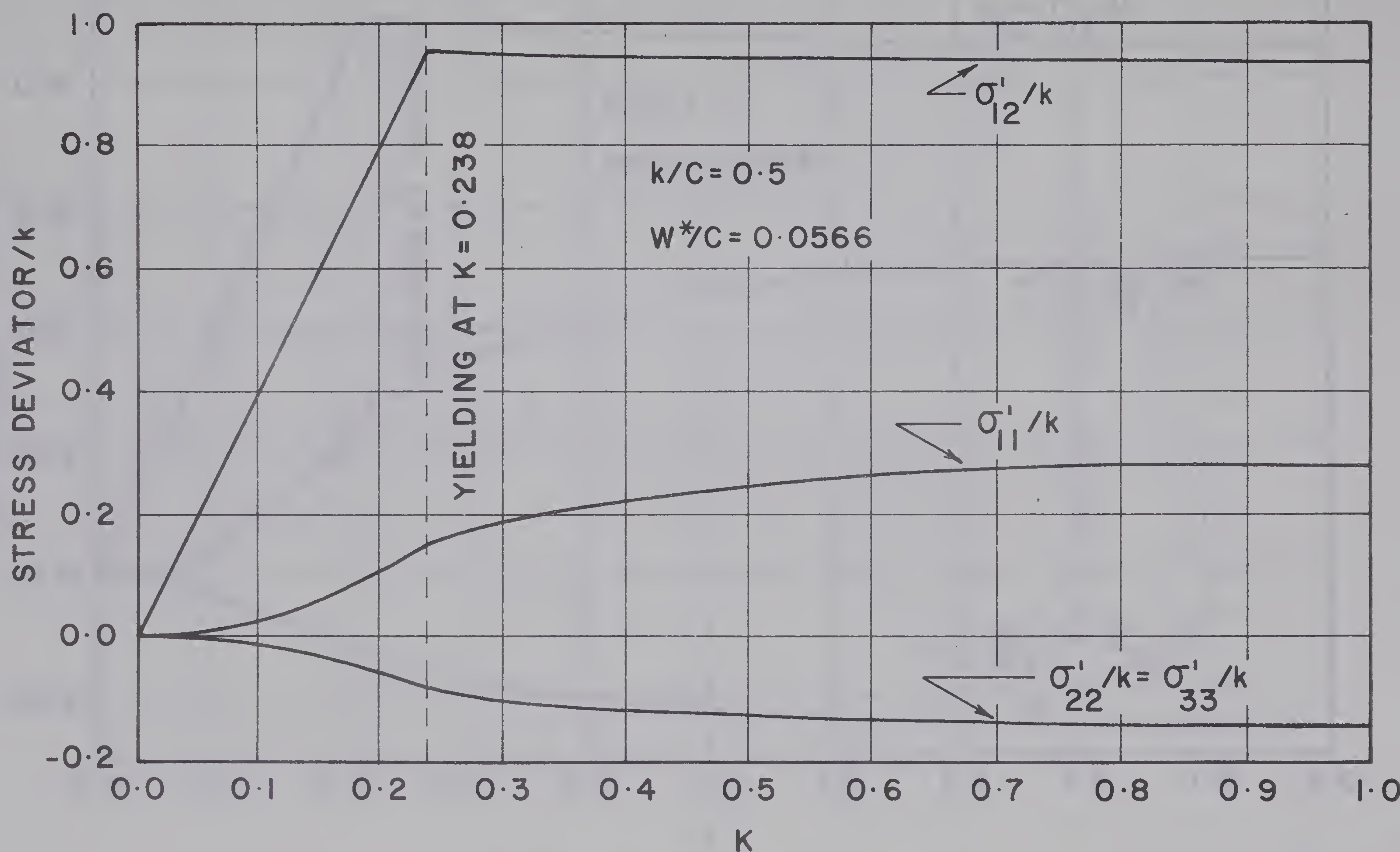



Figure 5.7 Maximum Shear Strain Energy Yield Condition,

$\sigma'_{ij}/k$ versus K

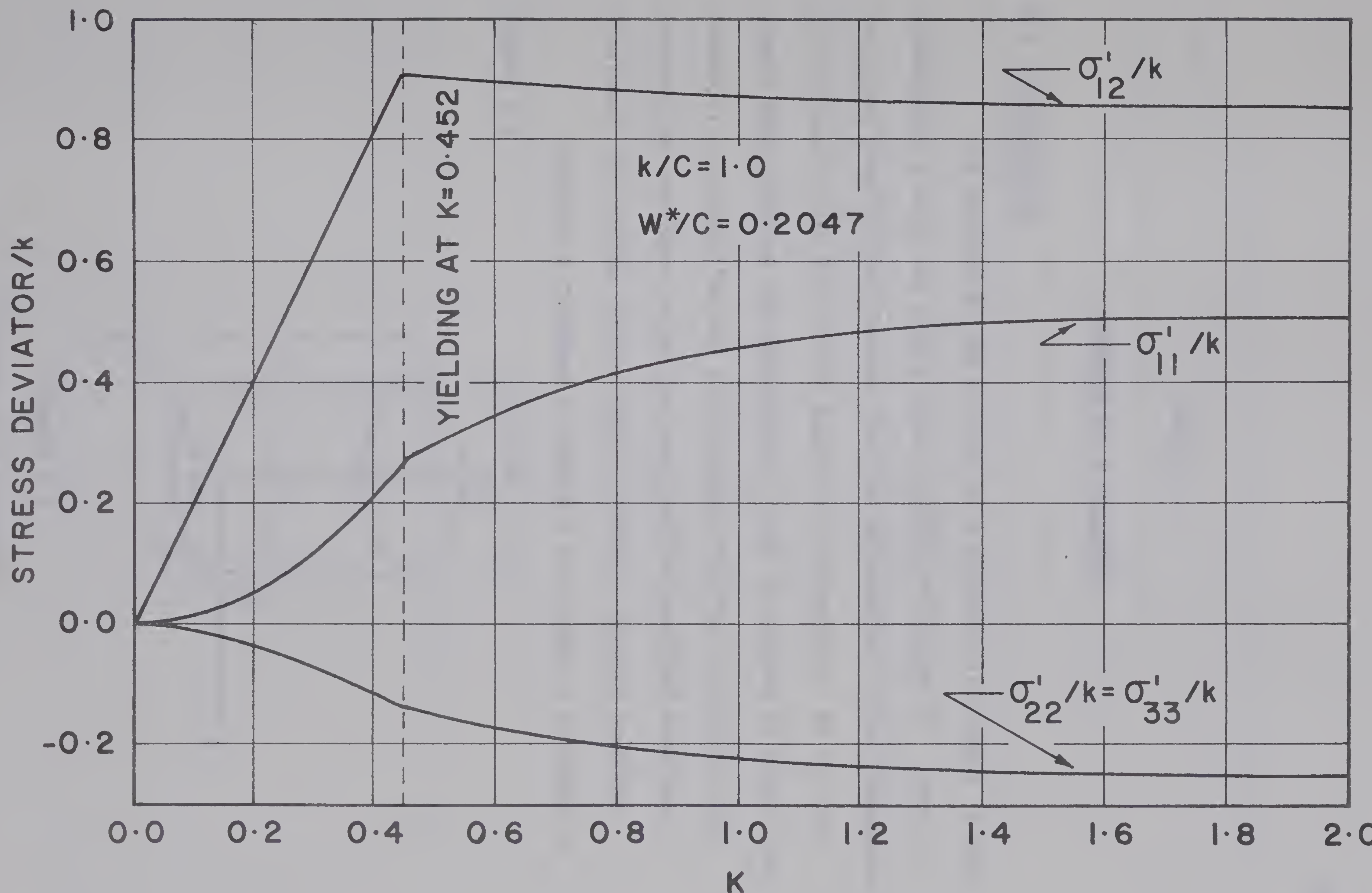



Figure 5.8 Maximum Shear Strain Energy Yield Condition,

$\sigma'_{ij}/k$ versus K

# CHAPTER VI

# TORSION OF A CYLINDER

## 6.1 Introduction

The problem of the torsion of a cylinder is closely related kinematically with the simple shear problem of Chapter V. There is the added complication however that the deformation is non-homogeneous so that use of the equilibrium equation is necessary. As for the simple shear problem the deformation, which is assumed to occur isothermally, is specified and the stresses required to maintain the deformation are found using the constitutive equations developed previously for finite strain elastic-perfectly plastic problems.

Consider a cylinder of length $\ell$ and radius a as shown in Figure 6.1.

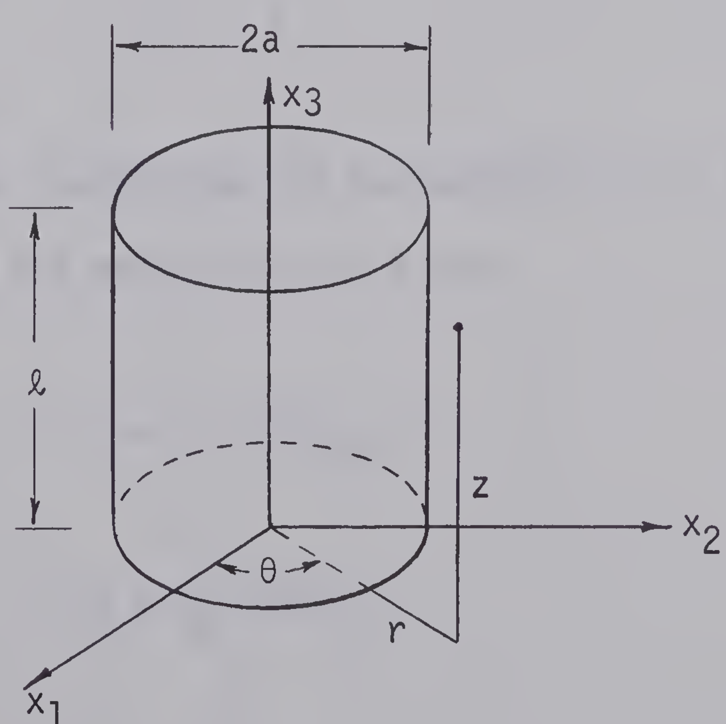



Figure 6.1

Torsion of a Cylinder

The $x_3$ axis of the fixed Cartesian coordinate system coincides with the axis of the cylinder and the coordinate system $(r,\theta,z)$ is a cylindrical polar coordinate system as shown.

The deformation is such that planes perpendicular to the axis of revolution remain plane and all material particles in any such plane move in concentric circles with centre on the axis of revolution. This is an isochoric non-homogeneous deformation.

Let $(\xi_1,\xi_2,\xi_3)$ be the coordinates of a particle referred to a convected coordinate system

$$\xi_1 = r \,,$$

$$\xi_2 = \theta \,,$$

and $$\xi_3 = z \,,$$

where r, $\theta$, and z are the coordinates of the particle in C.3. The Cartesian coordinates of the particle in C.3 are

$$x_1 = \xi_1 \cos \xi_2 \,,$$

$$x_2 = \xi_2 \sin \xi_2 \,,$$

and $$x_3 = \xi_3 \,. \tag{6.1.1}$$

Let $\phi$ be the angle of twist per unit length of the cylinder. The coordinates of the particle in C.1 referred to the Cartesian coordinate system are

$$X_1 = \xi_1 \cos(\xi_2 - \phi\xi_3) ,$$

$$X_2 = \xi_1 \sin(\xi_2 - \phi\xi_3) ,$$

and
$$X_3 = \xi_3 . \tag{6.1.2}$$

It follows from equations (3.2.5) and (3.2.6) that the metric tensors in C.1 and C.3, referred to the convected coordinate system have covariant and contravariant components

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\xi_1)^2 & -\phi(\xi_1)^2 \\ 0 & -\phi(\xi_1)^2 & 1+(\phi\xi_1)^2 \end{bmatrix}$$

$$[G^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi^2+(\frac{1}{\xi_1})^2 & \phi \\ 0 & \phi & 1 \end{bmatrix} \tag{6.1.3}$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\xi_1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

and

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/\xi_1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \qquad (6.1.4)$$

Consider a particle at o' in C.3 as shown in Figure 6.2. Let $ox_i'$ be a local Cartesian coordinate system with origin o' and with axes in the directions of the tangents to the convected coordinate

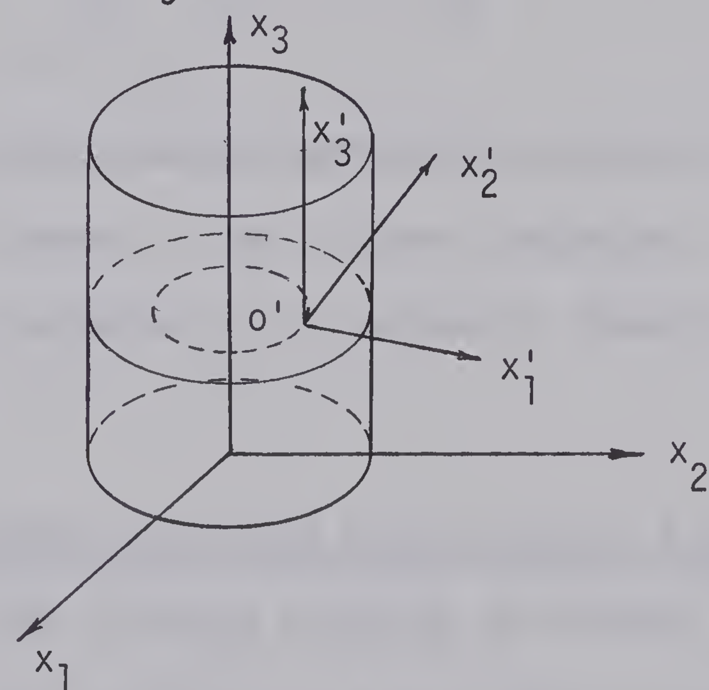



Figure 6.2

Local Cartesian Coordinate System

curves passing through o'. If $F_{iK}$ is the deformation gradient tensor at o' referred to the coordinate system $ox_i'$, it may be deduced from equation (3.2.5a) that $F_{iK}F_{jK}$ are the physical components of the metric tensor $G^{ij}$ at the point o'. These physical components are given by

$$G^{(ij)} = G^{ij} \sqrt{g_{\underline{ii}}} \sqrt{g_{\underline{jj}}} ,$$

where the bars under the repeated indices i and j indicate that there is no summation on these indices. Thus

$$[G^{(ij)}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+(\xi_1\phi)^2 & \xi_1\phi \\ 0 & \xi_1\phi & 1 \end{bmatrix} .$$

This indicates that for the deformation defined by equations (6.1.1) and (6.1.2) each material element in the cylinder undergoes a simple shear deformation with the parameter K, as defined in Chapter V, equal to $\xi_1\phi$.

6.2 <u>Stresses Prior to Yielding and in the Elastic Region After Yielding</u>

The solution for the stresses required to maintain a state of torsion in a right circular cylinder of a neo-Hookean material was first obtained by Rivlin [63]. This solution shows that a deformation given by equations (6.1.1) and (6.1.2) can be maintained by surface tractions on the ends of the cylinder without body forces or other surface tractions.

Consider a right circular cylinder composed of a neo-Hookean elastic-perfectly plastic material, subjected to the deformation described in section 6.1. Assume that yielding has occurred in the region $\alpha \leq \xi_1 \leq a$ and that for $0 \leq \xi_1 \leq \alpha$ the material remains elastic.

Since $K = \xi_1\phi$ it follows from equations (5.2.2) and (5.2.3) that

$$\alpha = \frac{1}{\phi}\left[-\frac{3}{2} + \frac{1}{2}\sqrt{9 + 3\left(\frac{k}{C}\right)^2}\right]^{1/2}$$

and

$$\alpha = \frac{1}{\phi}\sqrt{\frac{W^*}{C}}$$

for materials which obey the von Mises and the maximum shear strain energy yield conditions respectively.

In the elastic region the convected stress tensor is given by

$$\tau^{ij} = 2C\ G^{ij} + pg^{ij}$$

since the material is assumed to be neo-Hookean prior to yielding. From equations (6.1.3) and (6.1.4) this leads to

$$\left.\begin{aligned} \tau^{11} &= \tau^{33} = 2C+p\ , \\ \tau^{22} &= 2C\left[\phi^2 + \frac{1}{(\xi_1)^2}\right] + \frac{p}{(\xi_1)^2}\ , \\ \tau^{23} &= 2C\phi , \\ \text{and}\qquad \tau^{12} &= \tau^{13} = 0\ . \end{aligned}\right\} \quad (6.2.1)$$

Substitution of the stresses (6.2.1) into the equilibrium equations in cylindrical polar coordinates (Appendix B) gives

$$\frac{\partial p}{\partial \xi_1} - 2C\phi^2\xi_1 = 0 ,$$

$$\frac{\partial p}{\partial \xi_2} = 0 ,$$

and

$$\frac{\partial p}{\partial \xi_3} = 0 ,$$

from which

$$\frac{dp(\xi_1)}{d\xi_1} - 2C\phi^2\xi_1 = 0 .$$

Integration with respect to $\xi_1$ yields

$$p(\xi_1) = C\phi^2(\xi_1)^2 + c_1 ,$$

where the integration constant $c_1$ is determined from the condition that the radial stress $\tau^{11}$ is a continuous function of the radius. Let $\sigma^*$ be the as yet unknown radial stress at the elastic-plastic boundary $\xi_1 = \alpha$, and let $\sigma_{ij}$ be the stress tensor at a point referred to the local Cartesian coordinate system $ox_i'$ at that point. That is $\sigma_{ij}$ are the physical components of the convected stress tensor. Thus

$$\sigma_{11} = 2C+p$$

$$= 2C + C(\xi_1\phi)^2 + c_1$$

so that $$\sigma_{11}(\alpha) = \sigma^*$$

implies that $$c_1 = \sigma^* - 2C - C\alpha^2\phi^2 .$$

In the elastic region then, the stress solution in non-dimensionalized form is

$$\left.\begin{aligned}
\frac{\sigma_{11}}{k} &= \frac{C}{k}\left[\left(\frac{r}{a}\right)^2 - \left(\frac{\alpha}{a}\right)^2\right]\psi^2 + \frac{\sigma^*}{k} = \frac{\sigma_{33}}{k} , \\
\frac{\sigma_{22}}{k} &= \frac{3C}{k}\left(\frac{r}{a}\right)^2 \psi^2 - \frac{C}{k}\left(\frac{\alpha}{a}\right)^2 + \frac{\sigma^*}{k} , \\
\frac{\sigma_{23}}{k} &= \frac{2C}{k}\left(\frac{r}{a}\right)\psi, \\
\text{and} \qquad \frac{\sigma_{12}}{k} &= \frac{\sigma_{13}}{k} = 0 ,
\end{aligned}\right\} \quad (6.2.2)$$

where $\psi = a\phi$. The solution prior to yield is obtained from (6.2.2) by setting $\alpha = a$ and $\sigma^* = 0$ .

Let $N_e$ and $M_e$ be the total end force and twisting moment respectively acting of the ends of the cylinder in the elastic region

$0 \leq \xi_1 \leq \alpha$. Thus

$$N_e = \int_0^\alpha 2\pi\xi_1\sigma_{33}\, d\xi_1$$

and

$$M_e = \int_0^\alpha 2\pi\xi_1{}^2\sigma_{23} d\xi_1$$

so that substituting for $\sigma_{33}$ and $\sigma_{23}$ from equation (6.2.2), performing the integration and putting in non-dimensionalized form gives

$$\frac{N_e}{ka^2} = \pi[- \frac{C}{2k}\, \psi^2(\frac{\alpha}{a})^4 + \frac{\sigma^*}{k}\, (\frac{\alpha}{a})^2]$$

and

$$\frac{M_e}{ka^3} = \frac{\pi C\psi}{k}\, (\frac{\alpha}{a})^4 \ .$$

## 6.3 Stresses in the Elastic-Plastic Region

As shown in section 6.1, the deformation of each material element in the cylinder is equivalent to a simple shear deformation. Thus in the elastic-plastic region the stress deviator tensor $\sigma'_{ij}$ referred to the local Cartesian coordinate system is known in tabulated form from the simple shear solution found in Chapter V.

The total stress tensor is given by

$$\sigma_{ij} = \sigma'_{ij} + p\delta_{ij} \ ,$$

where $p = \frac{1}{3}\,\sigma_{kk}$ is the unknown mean normal stress which must be

determined from the equilibrium equations. This yields

$$\frac{\partial}{\partial \xi_1}(\sigma'_{11}+p) + \frac{1}{\xi_1}(\sigma'_{11}-\sigma'_{22}) = 0 \tag{6.3.1}$$

and
$$\frac{\partial p}{\partial \xi_2} = \frac{\partial p}{\partial \xi_3} = 0 \ .$$

Integration of equation (6.3.1) gives

$$\sigma'_{11} + p = -\int_{\delta}^{\xi_1} \frac{1}{\rho}[\sigma'_{11}(\rho) - \sigma'_{22}(\rho)]d\rho + c_2$$

and since the surface $\xi_1 = a$ is assumed to be stress free

$$c_2 = \int_{\delta}^{a} \frac{1}{\rho}[\sigma'_{11}(\rho) - \sigma'_{22}(\rho)]d\rho$$

and
$$p(\xi_1) = -\sigma'_{11}(\xi_1) + \int_{\xi_1}^{a} \frac{1}{\rho}[\sigma'_{11}(\rho) - \sigma'_{22}(\rho)]d\rho \ .$$

Thus in the elastic-plastic region the stress solution is obtained from

$$\left.\begin{aligned} \sigma_{11} &= \int_{\xi_1}^{a} \frac{1}{\rho}[\sigma'_{11}(\rho) - \sigma'_{22}(\rho)]d\rho = \sigma_{33} \ , \\ \sigma_{22} &= \sigma'_{22}-\sigma'_{11} + \int_{\xi_1}^{a} \frac{1}{\rho}[\sigma'_{11}(\rho) - \sigma'_{22}(\rho)]d\rho \ , \\ \sigma_{23} &= \sigma'_{23} \ . \end{aligned}\right\} \tag{6.3.2}$$

The constant $\sigma^*$ defined in section 6.2 is obtained from the above also and is

$$\sigma^* = \sigma_{11}\Big|_{\xi_1=\alpha} = \int_{\alpha}^{a} \frac{1}{\rho}\,[\sigma'_{11}(\rho) - \sigma'_{22}(\rho)]\,d\rho\ .$$

The total end force and twisting moment respectively, acting on the ends of the cylinder are given in non-dimensionalized form by

$$\frac{N}{ka^2} = \frac{N_e}{ka^2} + \int_{\alpha/a}^{\xi_1/a} 2\pi\left(\frac{\xi_1}{a}\right)\frac{\sigma_{33}}{k}\,d\left(\frac{\xi_1}{a}\right) \tag{6.3.3}$$

and

$$\frac{M}{ka^3} = \frac{M_e}{ka^3} + \int_{\alpha/a}^{\xi_1/a} 2\pi\left(\frac{\xi_1}{a}\right)^2\frac{\sigma_{23}}{k}\,d\left(\frac{\xi_1}{a}\right)\ . \tag{6.3.4}$$

## 6.4 Numerical Solution

The solution to this problem is necessarily a numerical one since it is based on the previously obtained numerical solution to the elastic-plastic simple shear problem. The latter solution gives the stress deviator tensor as a function of the parameter K and since for the torsion problem $K = (\frac{\xi_1}{a})\psi$ the integrals in equations (6.3.2) may be evaluated numerically using the tabulated values for $\sigma'_{ij}$ .

A computer program has been written to do the numerical integrations in equations (6.3.2), (6.3.3), and (6.3.4). The integrations are preformed using the sixth order Gaussian quadrature technique [67] described in Appendix D. The stress $\sigma_{ij}$ is determined at a finite

number of discrete points along a radius at intervals of $\Delta(\frac{\xi_1}{a})$ equal to 0.05. The computation proceeds from $\xi_1/a = 1$ to $\xi_1/a = \alpha/a$ so that at any of the computation points in the elastic-plastic region the integral in equation (6.3.2) is given as a sum of the previously calculated integral plus the integral over the last $\Delta(\xi_1/a)$ interval.

Since the solution for the stress deviator tensor which appears in the integrand is known only for discrete values of $K = (\xi_1/a)\psi$, it is necessary to perform an interpolation to determine the values at the intermediate values of K as required by the Gaussian quadrature routine. This interpolation is done using the fifth order Newton forward formula discussed in Appendix E.

The use of the Gaussian quadrature method for the integrations in equations (6.3.3) and (6.3.4) requires also that interpolations be made between the tabulated values of the stress solution $\sigma_{ij}$. It is not possible to use Newton's forward formula to perform this interpolation near $(\xi_1/a) = 1$ since the higher differences require stress points outside the actual cylinder. This difficulty may be overcome by using a backward difference formula but the additional problem arises that the interval $[\alpha/a,1]$ may not at times just after yielding contain enough stress points to allow the use of a fifth order interpolation formula. Since the stress solution is quite smooth, first, second, and third order interpolation formulae are used, with the first and second order formulae being used only when the number of stress points in the interval $[\alpha/a,1]$ prohibits the use of the third or

higher order formulae.

## 6.5 Discussion

Some of the numerical results obtained for both the von Mises and the maximum shear strain energy yield conditions are shown graphically in Figures 6.3 to 6.15. The solutions have been found for values of $k/C$ equal to 0.1, 0.5 and 1.0 and for the equivalent values of $W^*/C$ with $\psi$ taking the values 0.75 and 1.5.

The shapes of the stress/k versus r/a curves are of course quite similar to the shapes of the stress deviator/k versus K curves of the simple shear solution.

For values of $k/C \ll 1$ the stress solution is that of the classical small strain elastic-perfectly plastic theory.

As the deformation parameter $\psi$ reaches values of two or three times that of $k/C$, the normal end force and the twisting moment approach constant values and ultimately they become independent of subsequent twisting as does the twisting moment in the classical theory when almost the entire cylinder has yielded.

It is also observed that as $k/C$ goes from 0.1 to 1.0 the resulting change in the value of $M/ka^3$ for large $\psi$ is only of the order of magnitude of 0.1 whereas the resulting change in the value of $N/ka^2$ for large $\psi$ is of the order of magnitude of 10. That is, as in the simple shear problem it is the normal stress effects which change radically as the value of $k/C$ increases.

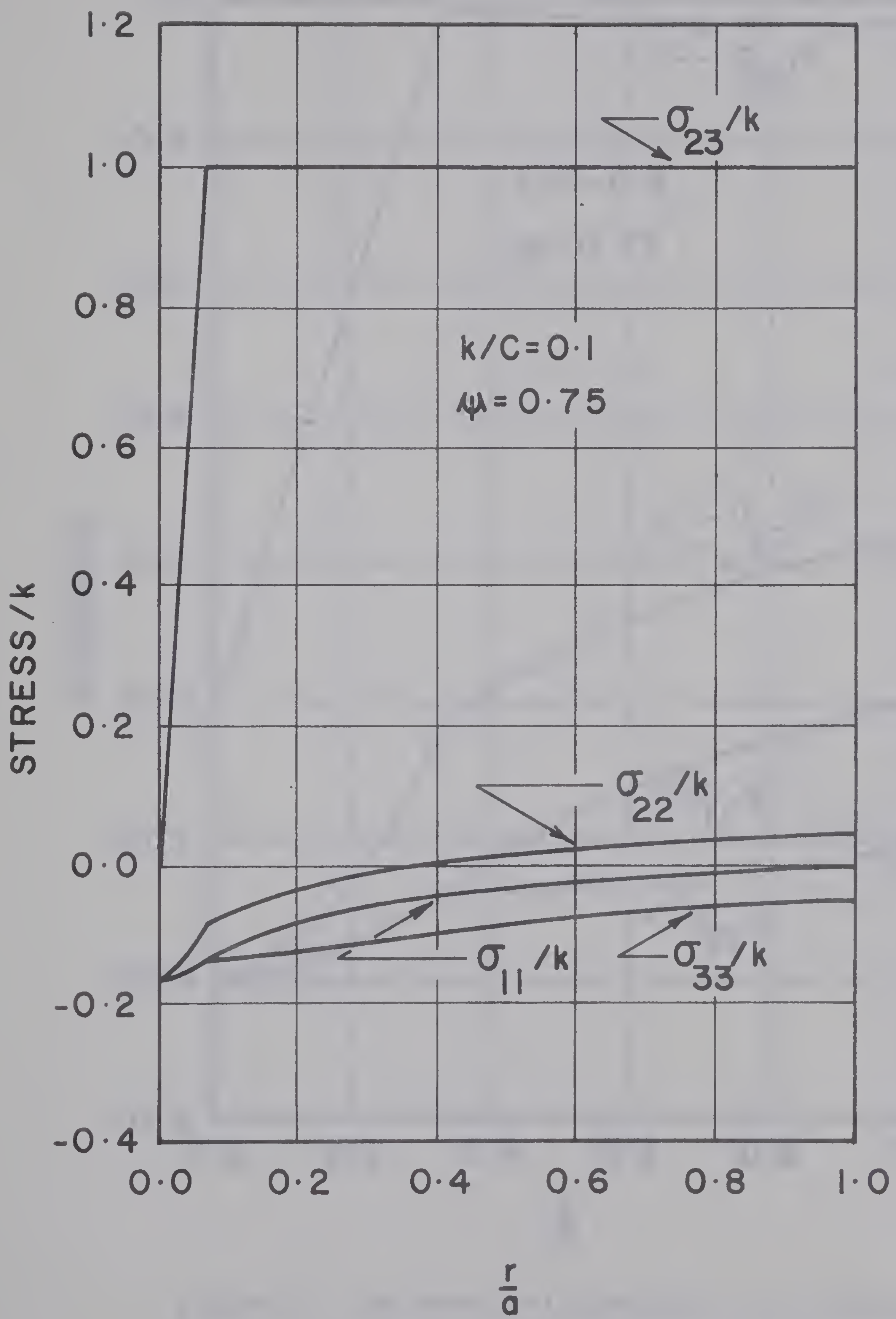



Figure 6.3 von Mises Yield Condition, $\sigma_{ij}/k$ versus $\frac{r}{a}$

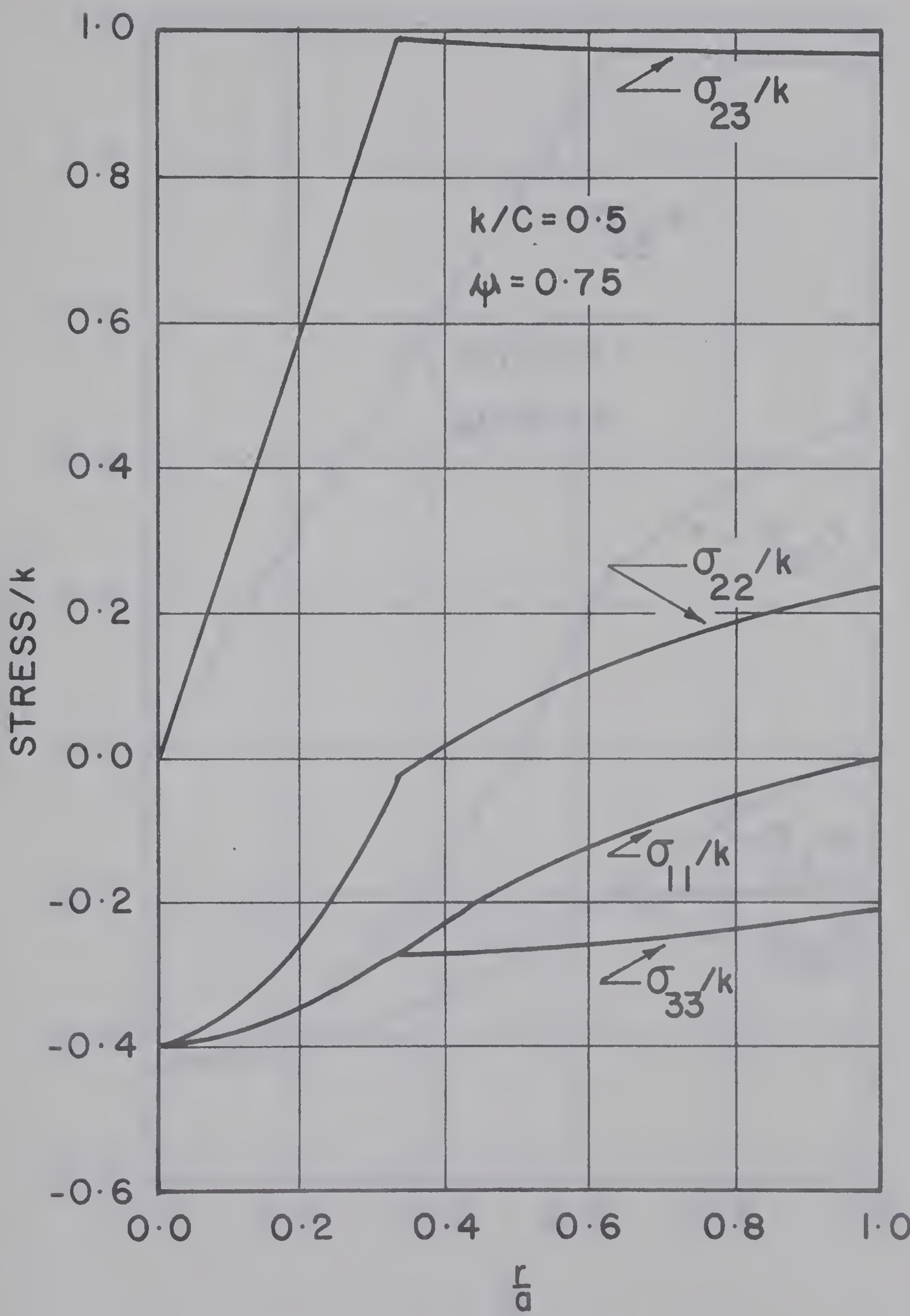



Figure 6.4 von Mises Yield Condition, $\sigma_{ij}/k$ versus $\frac{r}{a}$

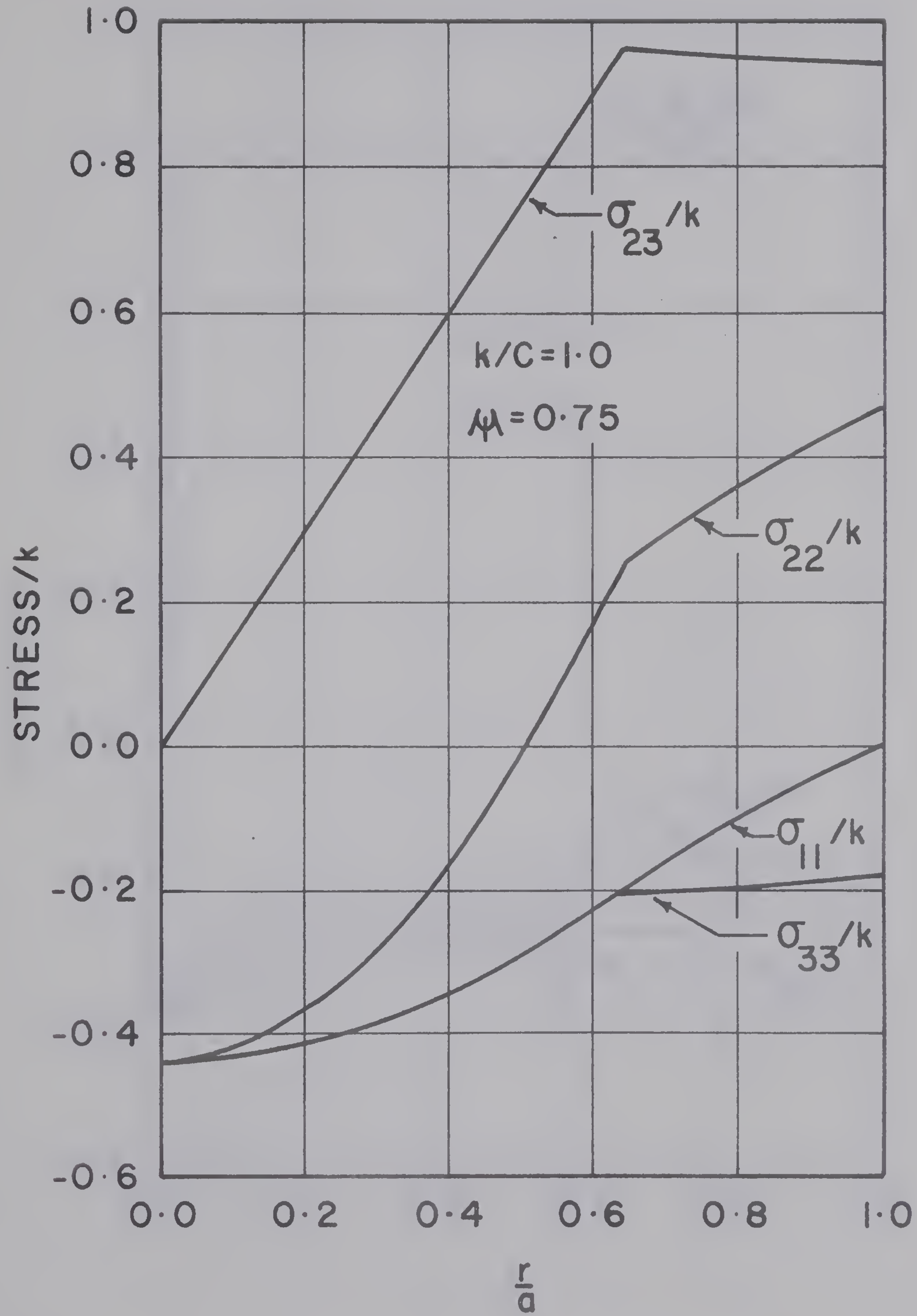



Figure 6.5 von Mises Yield Condition, $\sigma_{ij}/k$ versus $\frac{r}{a}$

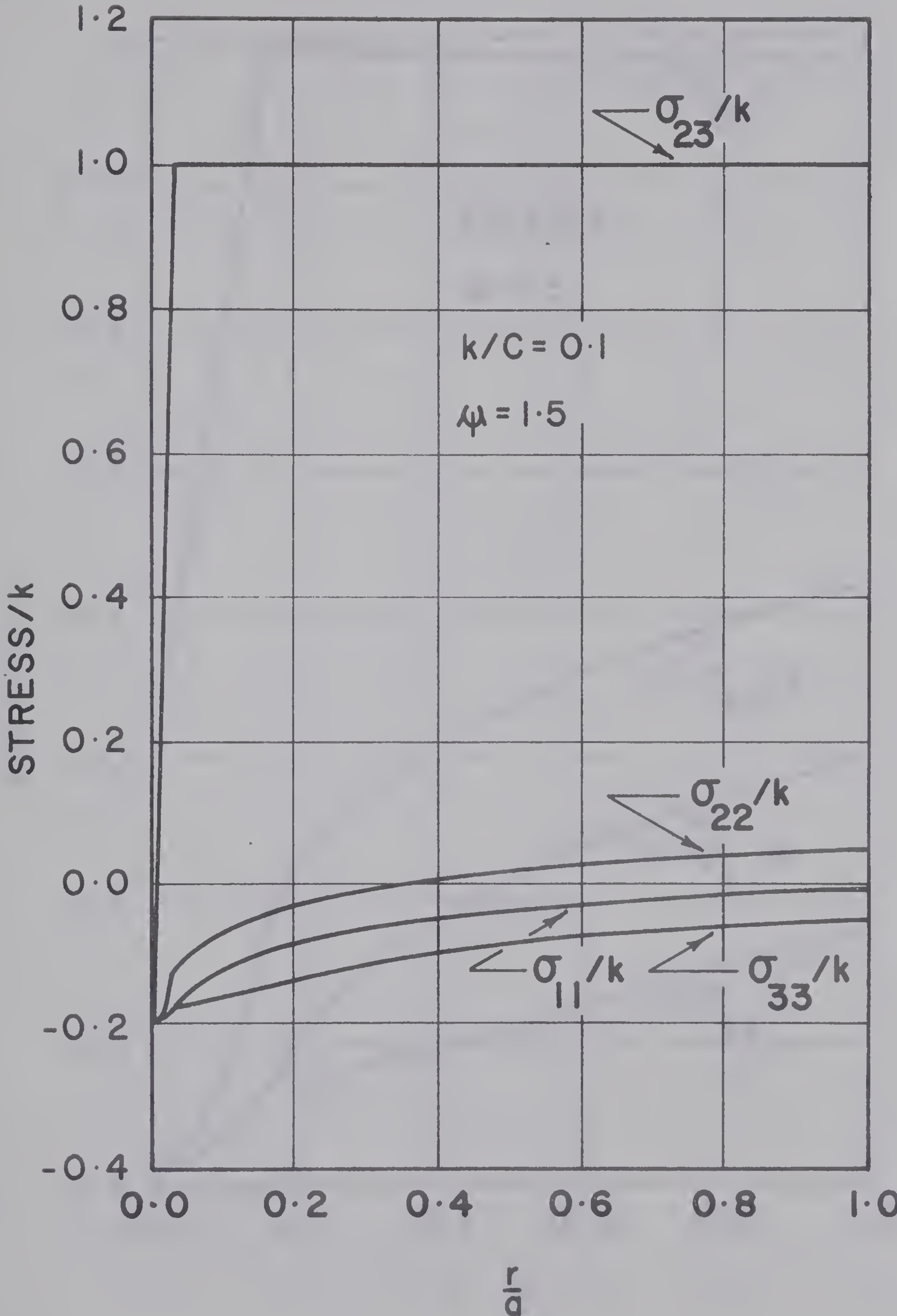



Figure 6.6 von Mises Yield Condition, $\sigma_{ij}/k$ versus $\frac{r}{a}$

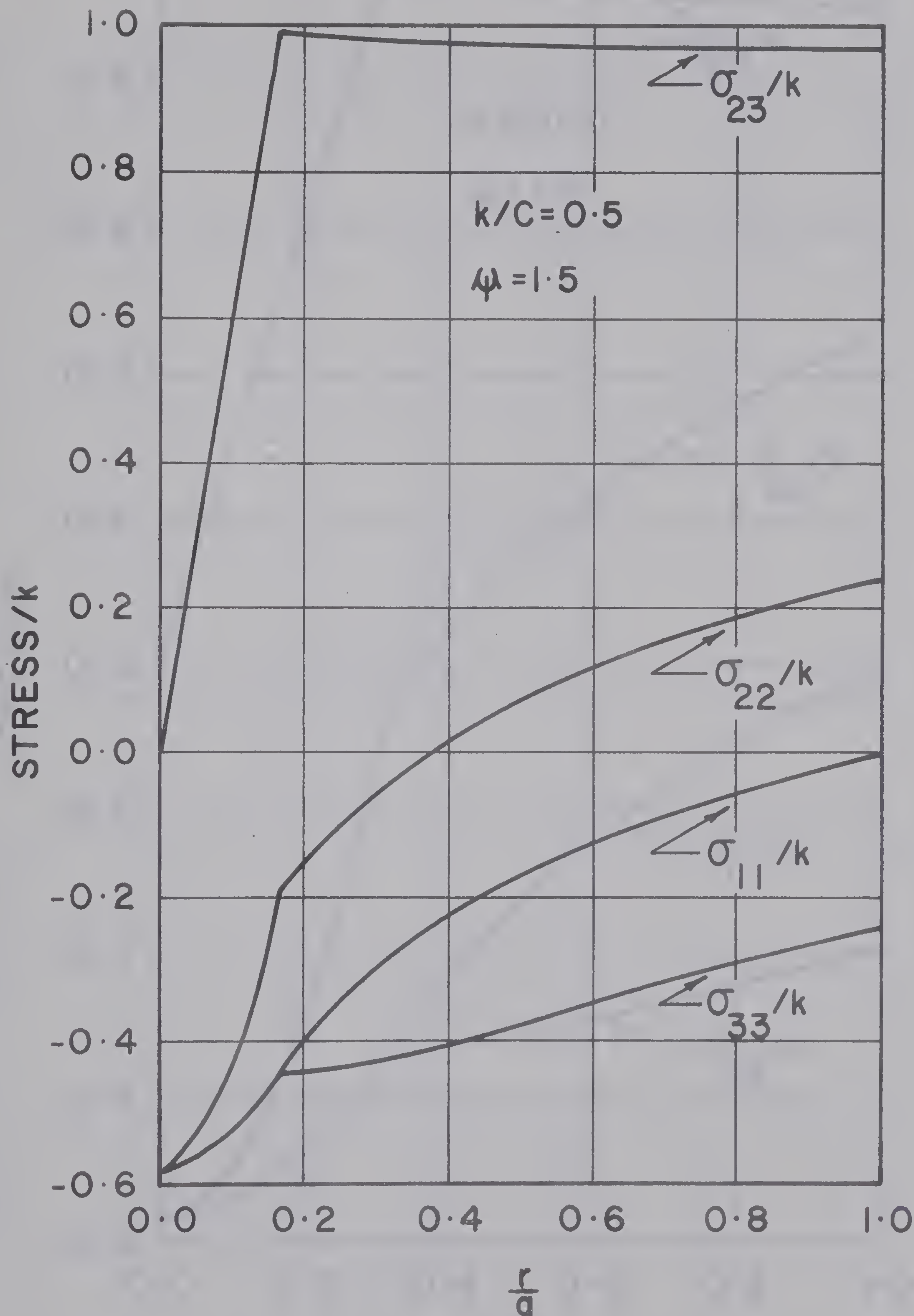



Figure 6.7 von Mises Yield Condition, $\sigma_{ij}/k$ versus $\frac{r}{a}$

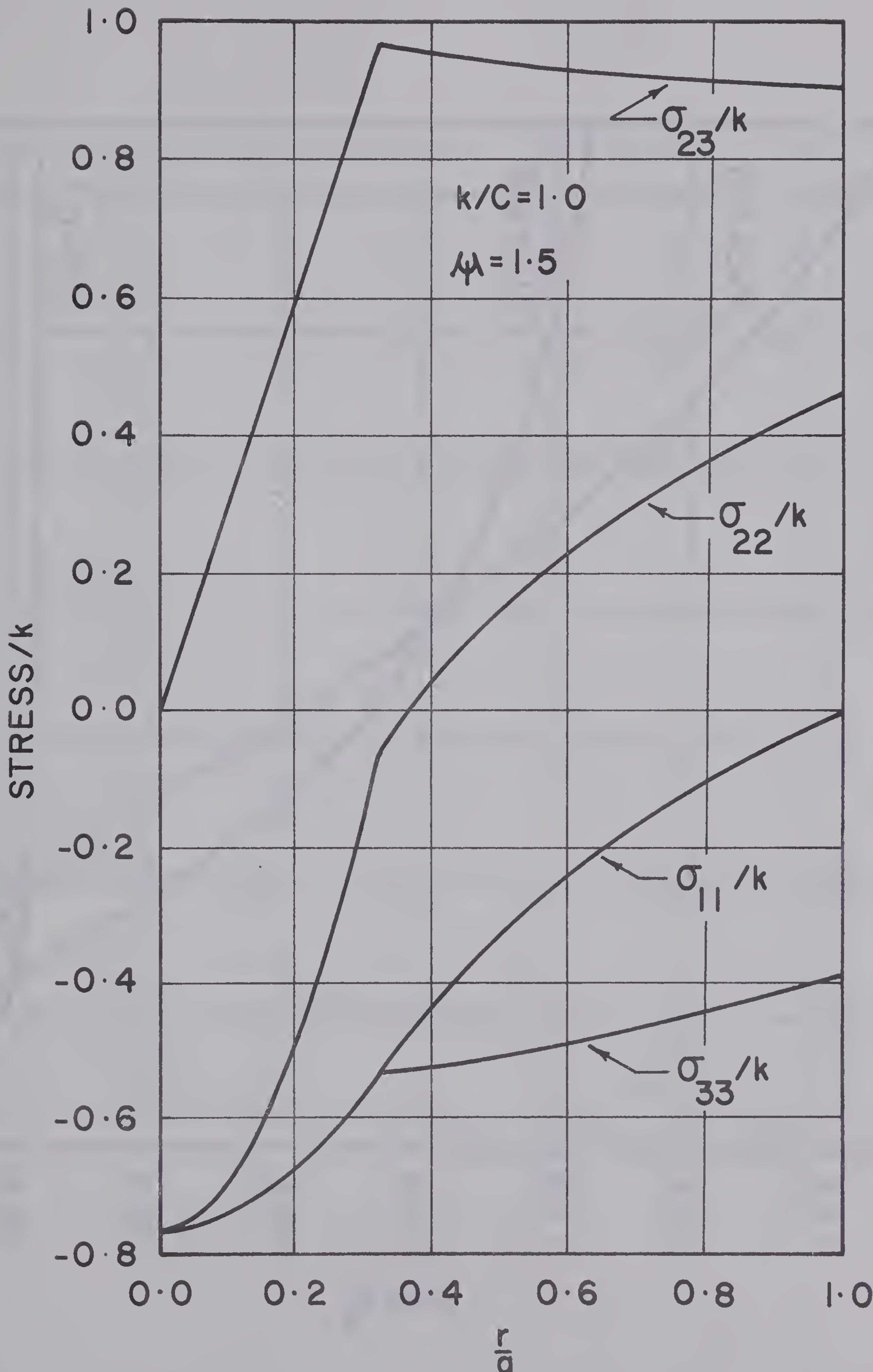



Figure 6.8 von Mises Yield Condition, $\sigma_{ij}/k$ versus $\frac{r}{a}$

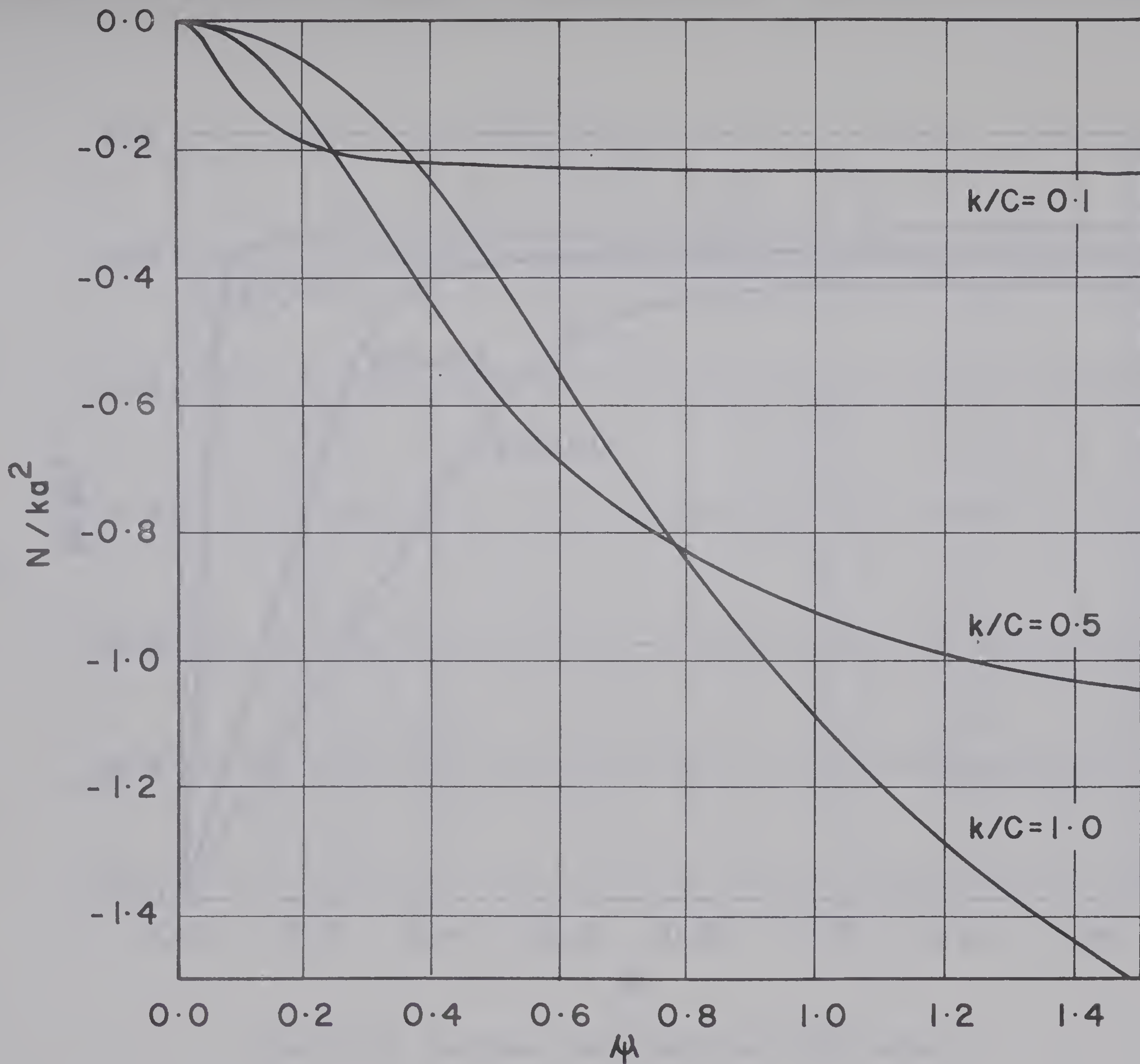
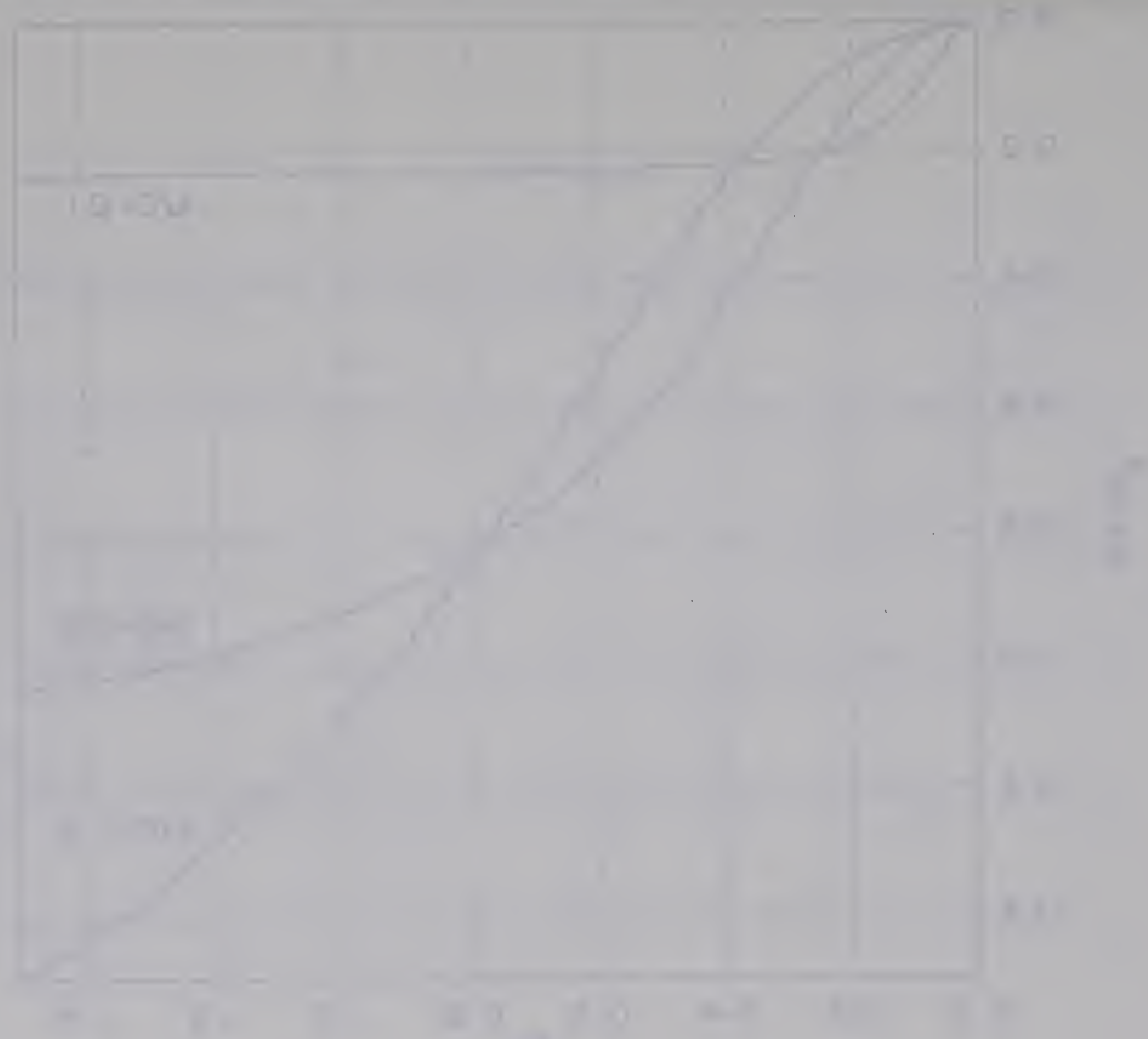



Figure 6.9 von Mises Yield Condition, N/ka$^2$ versus $\psi$



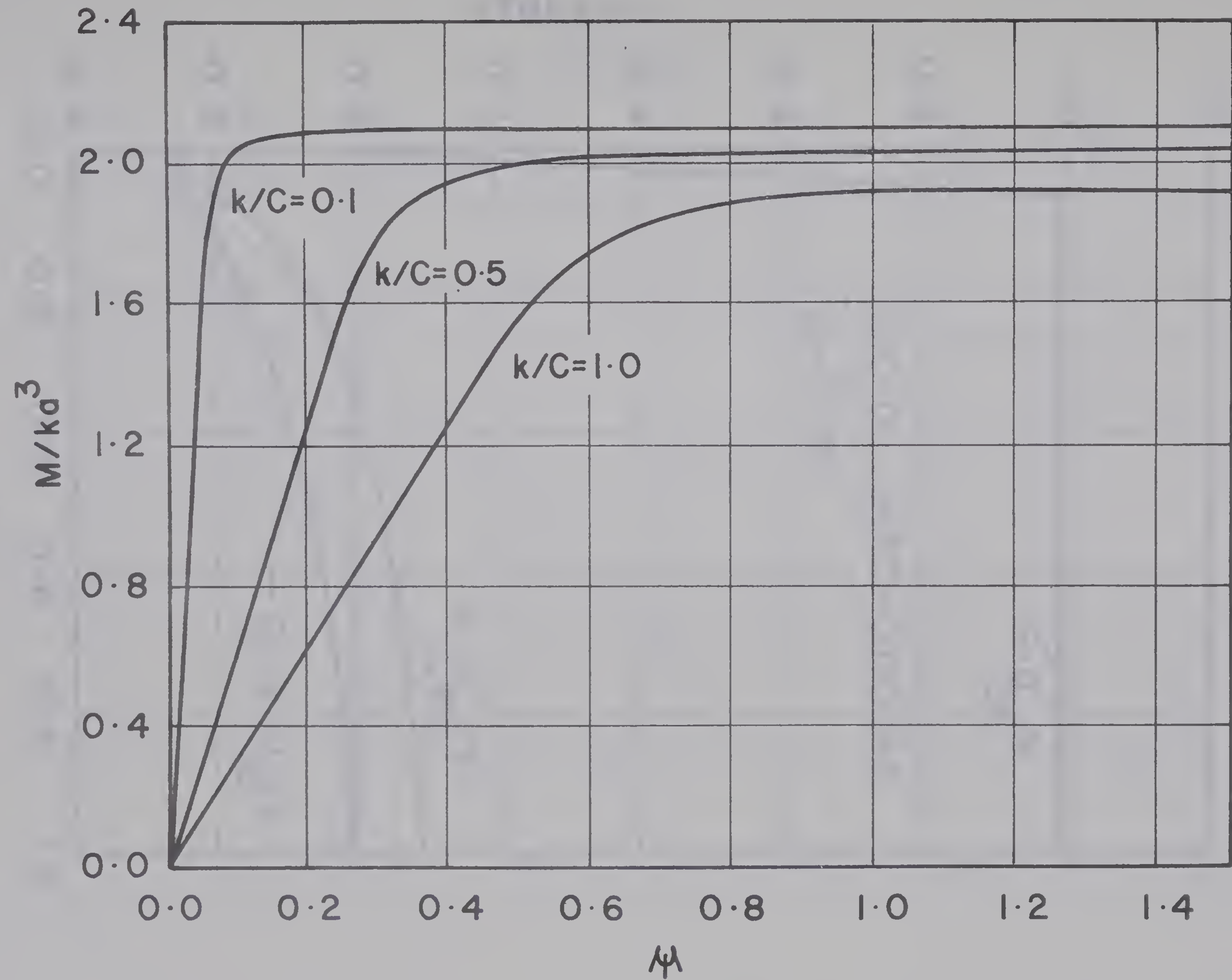



Figure 6.10 von Mises Yield Condition, $M/ka^3$ versus $\psi$

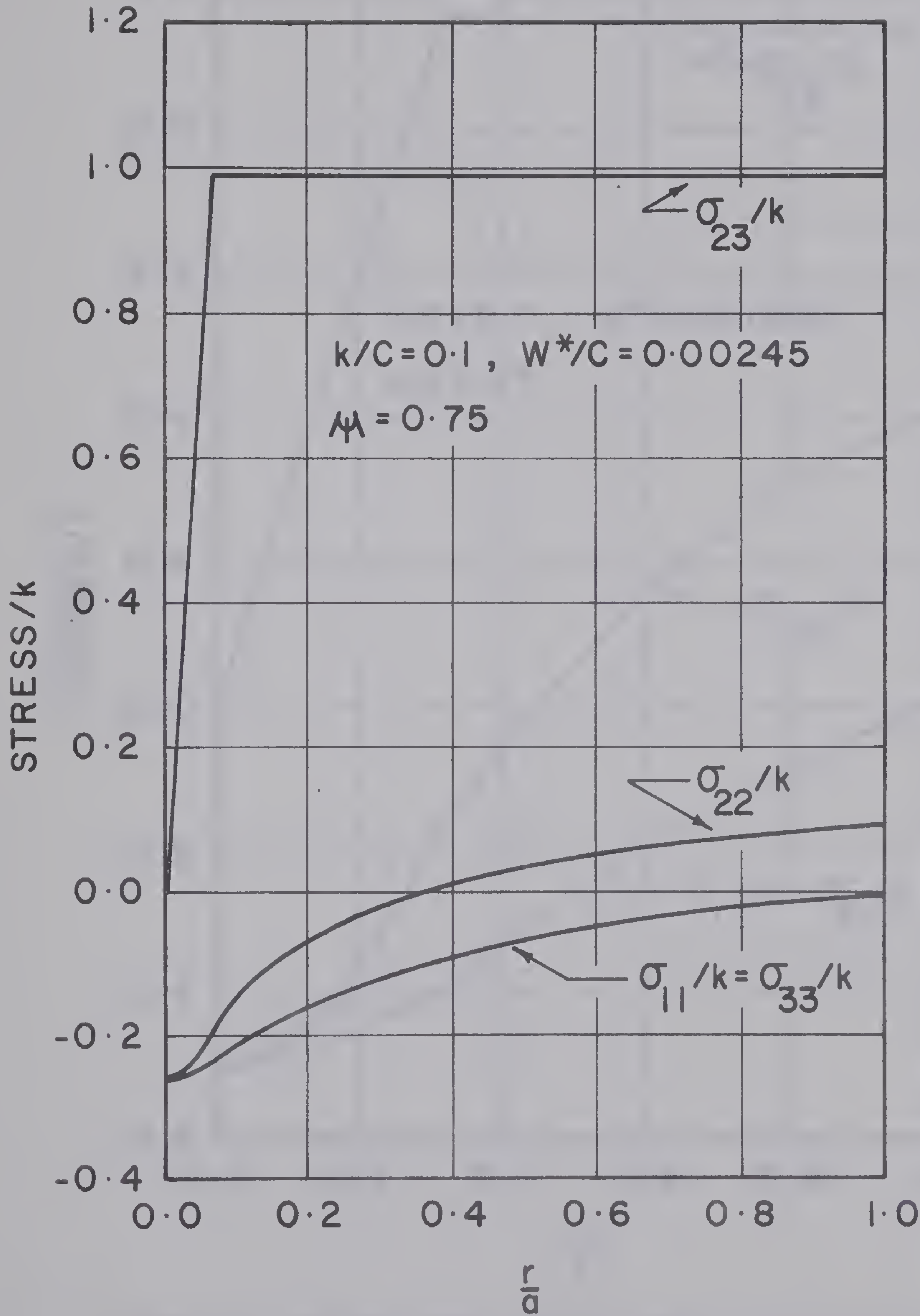



Figure 6.11 Maximum Shear Strain Energy Yield Condition,

$\sigma'_{ij}/k$ versus $\frac{r}{a}$

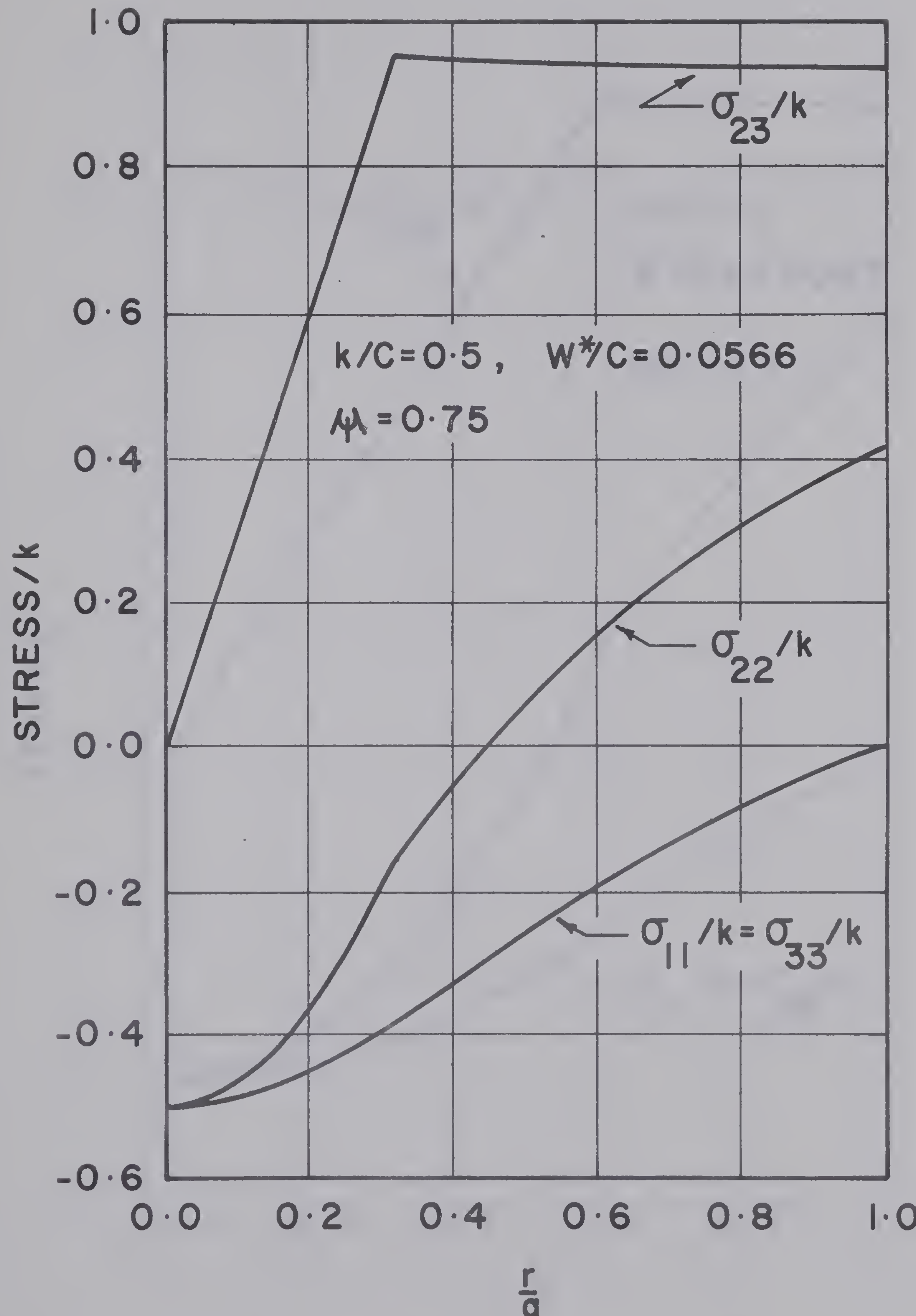



Figure 6.12 Maximum Shear Strain Energy Yield Condition, $\sigma'_{ij}/k$ versus $\frac{r}{a}$

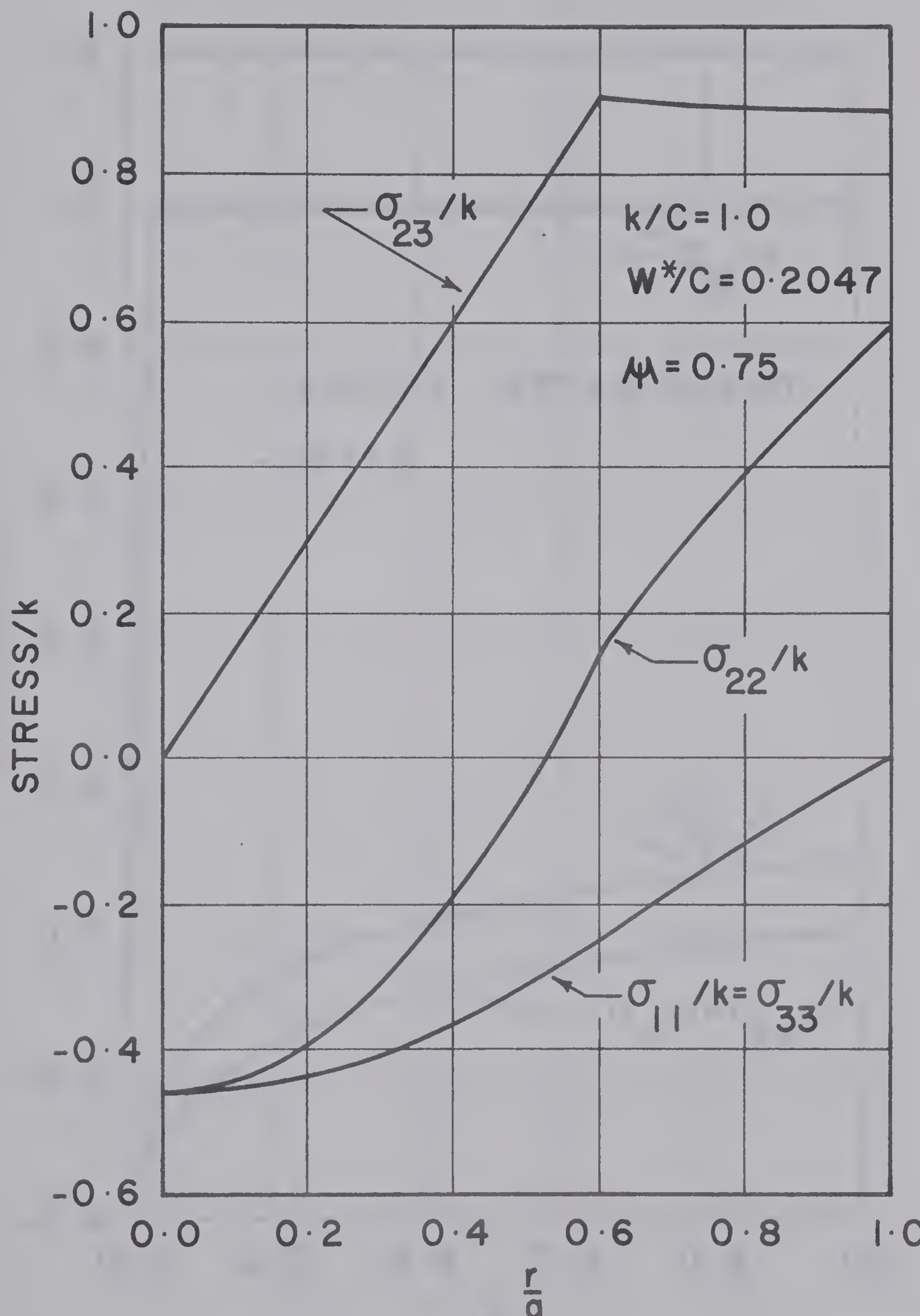



Figure 6.13 Maximum Shear Strain Energy Yield Condition, $\sigma'_{ij}/k$ versus $\frac{r}{a}$

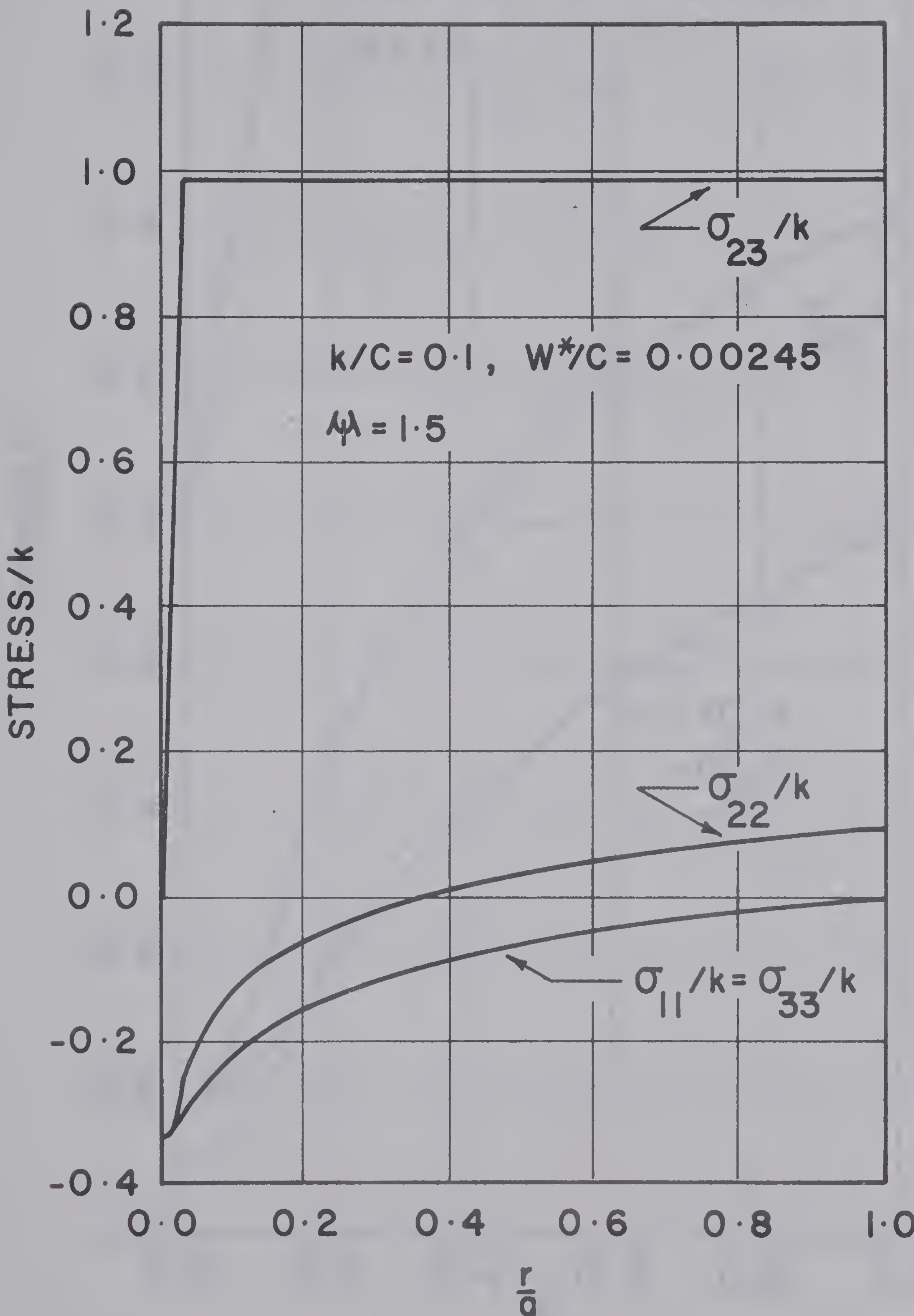



Figure 6.14 Maximum Shear Strain Energy Yield Condition, $\sigma_{ij}/k$ versus $\frac{r}{a}$

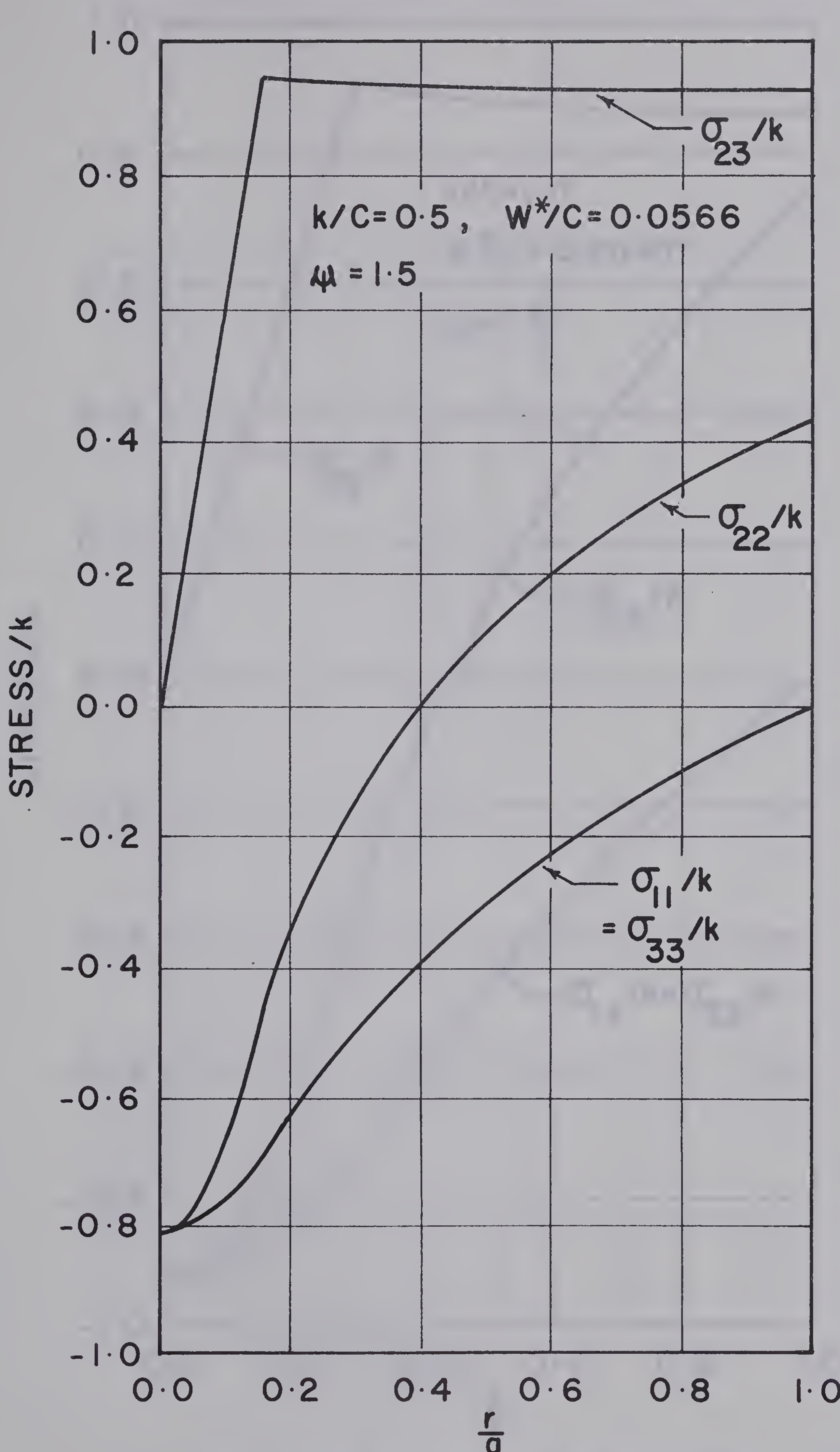



Figure 6.15 Maximum Shear Strain Energy

Yield Condition $\sigma_{ij}/k$ versus r/a

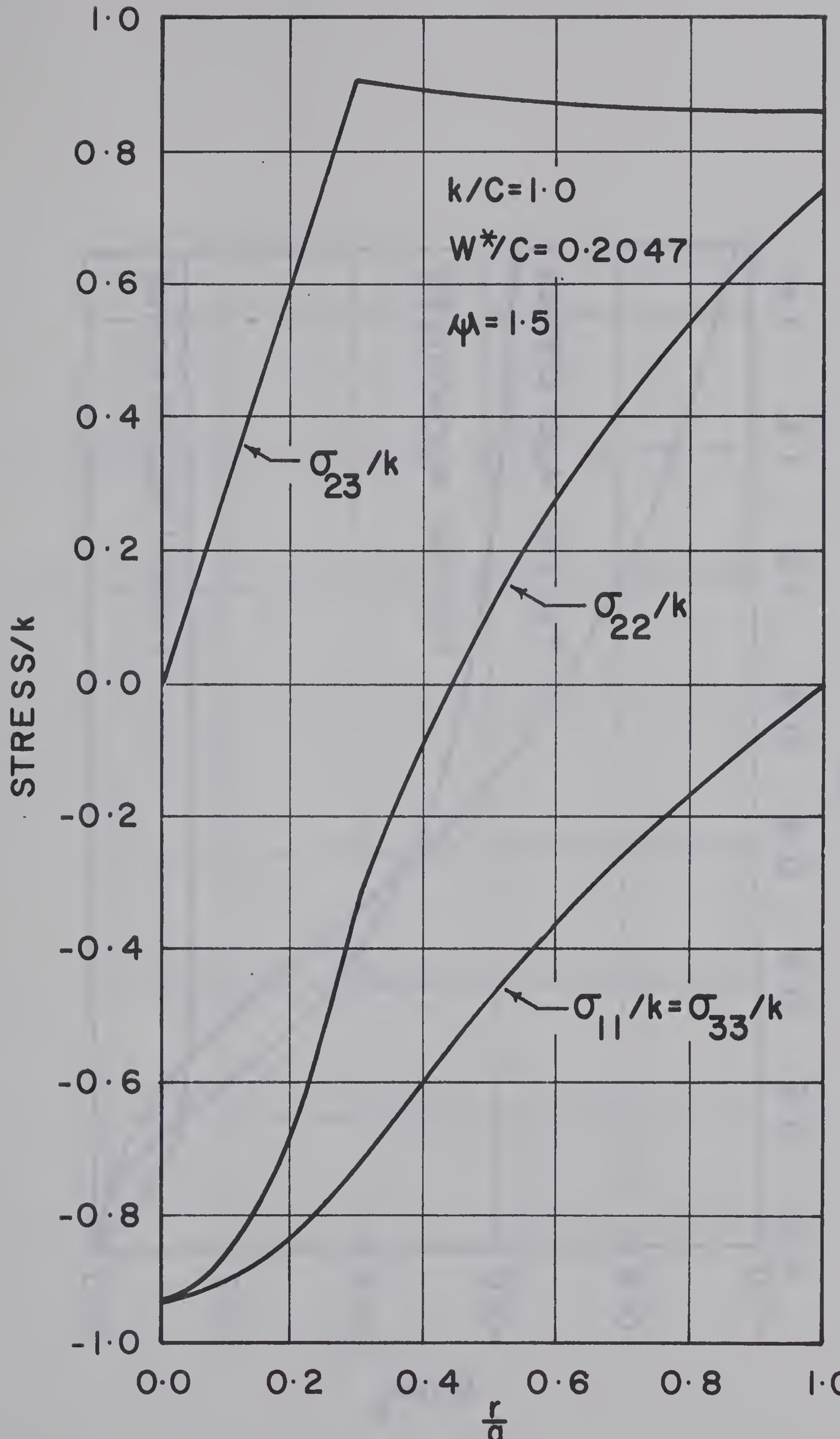



Figure 6.16 Maximum Shear Strain Energy Yield Condition, $\sigma_{ij}/k$ versus r/a

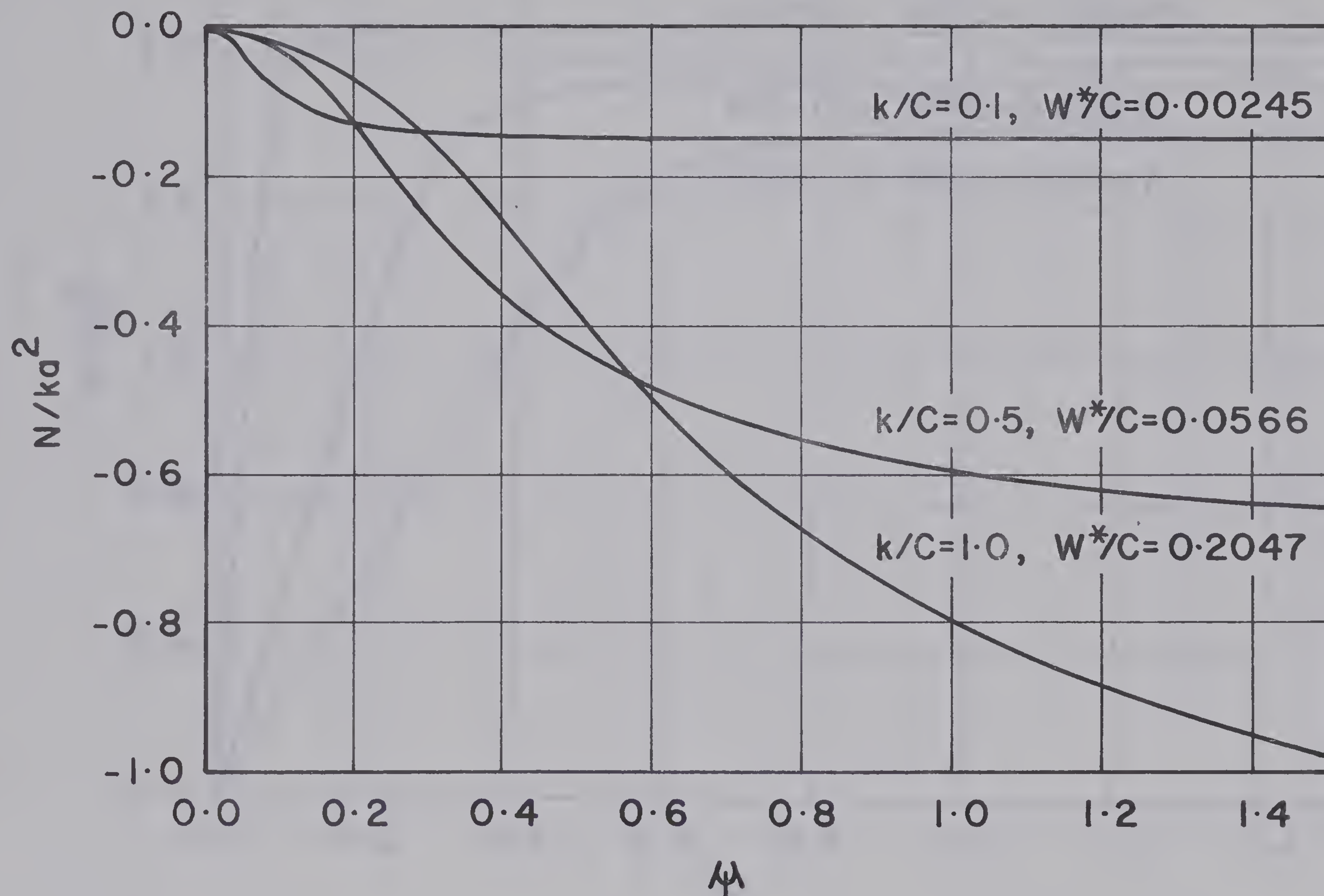



Figure 6.17 Maximum Shear Strain Energy Yield Condition, $N/ka^2$ versus $\psi$

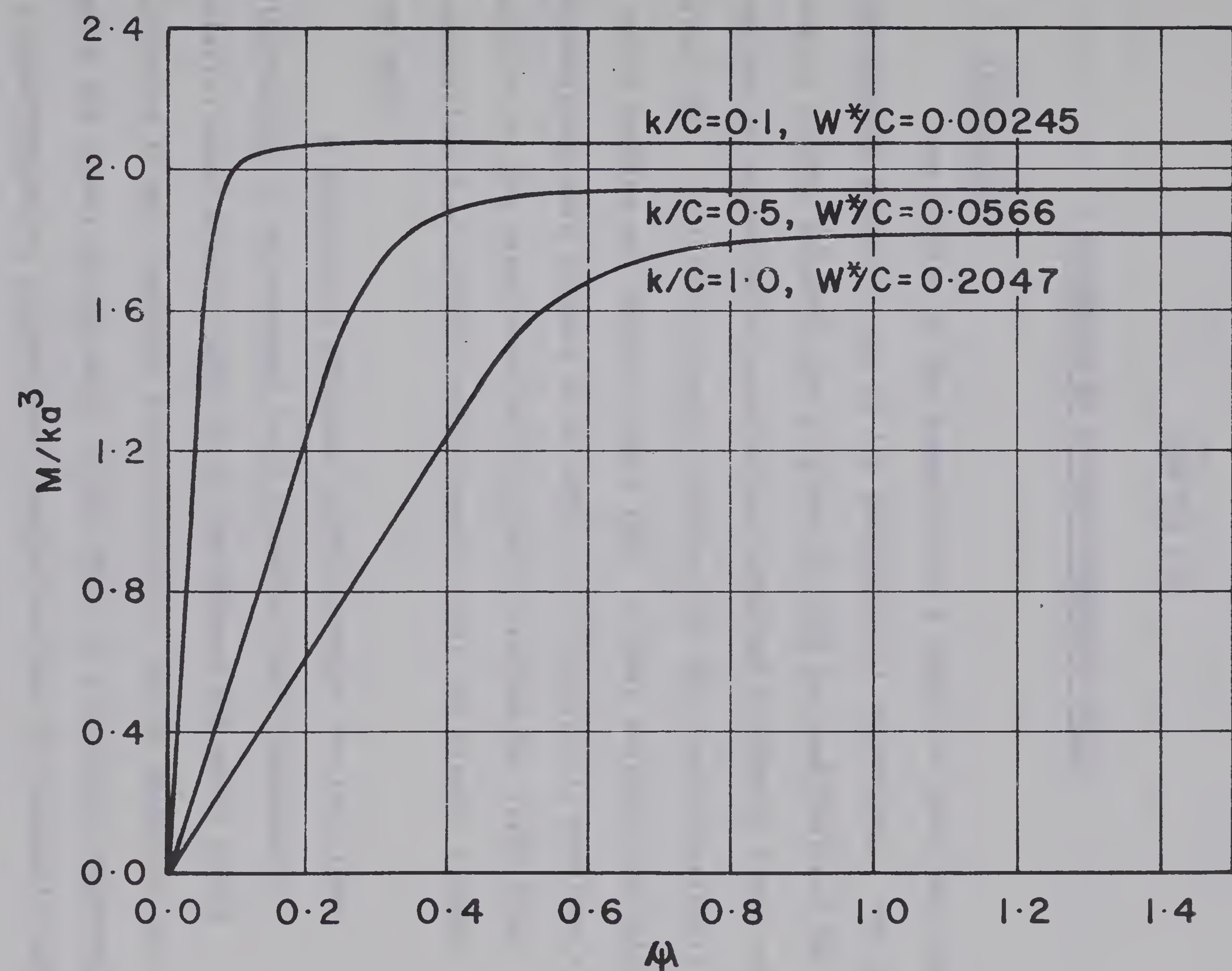



Figure 6.18 Maximum Shear Strain Energy Yield Condition, $M/ka^3$ versus $\psi$

# CHAPTER VII

# EXPANSION OF A THICK SPHERICAL SHELL

## 7.1 Introduction

The problem of the expansion of a spherical shell has received considerable attention due to its technological importance. The classical Hookean elastic solution was obtained in 1852 by Lamé [68] and the solution for an hyperelastic material was obtained in 1950 by Green and Shield [69]. The first correct solution for the elastic-perfectly plastic problem was found by Reuss [5]. In that analysis the total deformations were assumed to be small. Hill [70], [71] obtained a solution to the elastic-perfectly plastic problem for large total deformations but retained the assumption that the elastic strains are small.

A solution is obtained in this chapter for the stress distribution in an expanded thick spherical shell composed of a material whose elastic behavior is neo-Hookean and which yields according to any isotropic yield condition. Due to spherical symmetry the stress at any point in the shell is a uniaxial compression $-\sigma$ superimposed on a hydrostatic tension so that all isotropic yield conditions become $\sigma = Y$ where $Y$ is the uniaxial yield stress.

Since the deformation of the sphere is non-homogeneous, residual stresses result from unloading from an elastic-plastic state.

A solution for the residual stresses is also found.

The assumption of incompressibility greatly simplifies the problem. In the elastic-plastic region of the shell, the yield condition and the equilibrium equation are sufficient to determine the stress solution and it is not necessary to consider the plastic flow rule developed in Chapter IV.

## 7.2 Preliminary

Referring to Figure 7.1 let $(r,\theta,\phi)$ be a spherical polar coordinate system and $ox_i$ a Cartesian coordinate system.

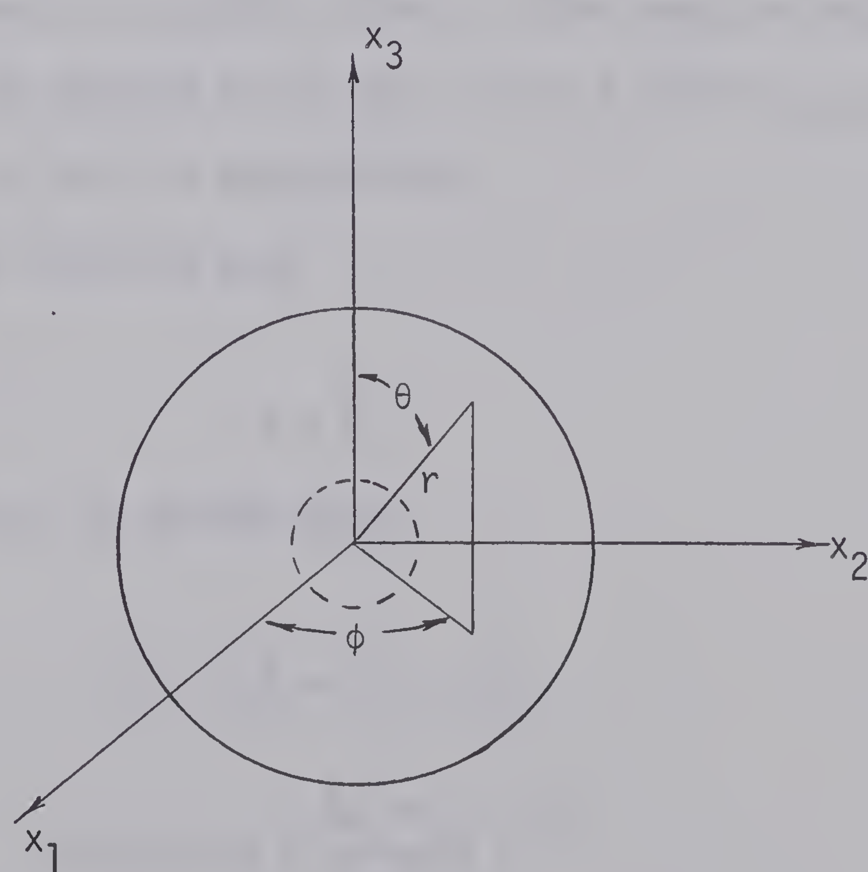



Figure 7.1

Thick Spherical Shell and

Spherical Polar Coordinate System

Four configurations of the shell, C.1, C.2, C.3, and C.4, as defined in section 2.1, are considered.

For a given state of deformation C.3, the solution if no yielding has occurred, is that found by Green and Shield [69]. Since the stresses are highest at the inner surface, it is assumed that yielding begins there. After the onset of yielding let a be the radius of the elastic-plastic boundary which is spherical due to symmetry and let the radii of this material surface in C.1 and C.4 be A and $\alpha$ respectively. Similarly let R and $\imath$ be the radii of the spherical surfaces in C.1 and C.4 which contain the same material particles as the spherical surface in C.3 with radius r. The inner and outer radii of the shell are then denoted by $(R_1,R_2)$, $(r_1,r_2)$ and $(\imath_1,\imath_2)$ in configurations C.1, C.3, and C.4 respectively.

Define the function Q by

$$Q = \frac{R}{r} .$$

From incompressibility it follows that

$$r^3 - r_1{}^3 = R^3 - R_1{}^3$$

and

$$Q(r) = [1 + \frac{R_1{}^3 - r_1{}^3}{r^3}]^{1/3} \tag{7.2.1}$$

The value of Q at $r = r_1$ is denoted by $Q_1$ and is a measure of the total deformation of the shell in C.3.

Let $(\xi_1, \xi_2, \xi_3)$ be the coordinates of a particle referred to a convected coordinate system [48] defined by

$$\xi_1 = r \ ,$$

$$\xi_2 = \theta \ ,$$

$$\xi_3 = \phi \ ,$$

where $(r, \theta, \phi)$ are the coordinates of the particle in C.3. The covariant and contravariant components of the metric tensor referred to the convected coordinate system are $(G_{ij}, G^{ij})$, $(\mathcal{G}_{ij}, \mathcal{G}^{ij})$, $(g_{ij}, g^{ij})$ and $(\mathscr{g}_{ij}, \mathscr{g}^{ij})$ in configurations C.1, C.2, C.3, and C.4 respectively.

Referred to the Cartesian coordinate system the coordinates of a particle in C.3 are

$$x_1 = \xi_1 \sin \xi_2 \cos \xi_3 \ ,$$

$$x_2 = \xi_1 \sin \xi_2 \sin \xi_3 \ ,$$

$$x_3 = \xi_1 \cos \xi_2 \ ,$$

and in C.1 the coordinates are

$$X_1 = \xi_1 Q \sin \xi_2 \cos \xi_3 \ ,$$

$$X_2 = \xi_1 Q \sin \xi_2 \sin \xi_3 ,$$

$$X_3 = \xi_1 Q \cos \xi_2 .$$

From equations (3.2.5) and (3.2.6) and the equation

$$\frac{dQ}{dr} = \frac{1}{r} \left(\frac{1}{Q^2} - Q\right) , \tag{7.2.2}$$

the contravariant components of the metric tensor in C.1 and C.3 are found to be

$$[G^{ij}] = \begin{bmatrix} Q^4 & 0 & 0 \\ 0 & \frac{1}{r^2 Q^2} & 0 \\ 0 & 0 & \frac{1}{r^2 Q^2 \sin^2 \theta} \end{bmatrix} \tag{7.2.3}$$

and

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{bmatrix} . \tag{7.2.4}$$

## 7.3 Elastic Region

Let $\tau^{ij}$ be the components of the stress tensor in C.3 referred to the convected coordinate system. The stress in the elastic region $a \leq r \leq r_1$ is determined from the constitutive equation

$$\tau^{ij} = 2CG^{ij} + pg^{ij} .$$

From equations (7.2.3) and (7.2.4) this gives

$$\tau^{11} = 2CQ^4 + p , \tag{7.3.1}$$

$$\tau^{22} = \frac{2C}{r^2Q^2} + \frac{p}{r^2} = \sin^2\theta\tau^{33} , \tag{7.3.2}$$

and
$$\tau^{12} = \tau^{13} = \tau^{23} = 0 . \tag{7.3.3a,b,c}$$

The equilibrium equations (Appendix C) are

$$\tau^{11}{}_{,1} - r\,\tau^{22} - r\sin^2\theta\tau^{33} + \frac{2}{r}\tau^{11} = 0 , \tag{7.3.4}$$

$$\tau^{22}{}_{,2} - \sin\theta\cos\theta\tau^{33} + \cot\theta\tau^{22} = 0 , \tag{7.3.5}$$

and
$$\tau^{33}{}_{,3} = 0 . \tag{7.3.6}$$

Substitution for the stresses in equation (7.3.4) gives

$$\frac{dp}{dr} = \frac{4C}{r}\left[\frac{Q^6-2Q^3+1}{Q^2}\right] . \tag{7.3.7}$$

Equations (7.3.5) and (7.3.6) are satisfied identically due to symmetry.

Using equation (7.2.2), the first equilibrium equation (7.3.7)

becomes

$$\frac{dp}{dQ} = 4C(1-Q^3) ,$$

integration of which gives

$$p = - CQ^4 + 4CQ + c_3 , \tag{7.3.8}$$

where $c_3$ is a constant of integration.

It is further assumed that the outer surface of the shell is stress free so that $\tau^{11} = 0$ at $r = r_2$.

Letting

$$Q_2 = Q(r_2)$$

it follows from (7.3.1) that

$$2CQ_2{}^4 + p(r_2) = 0$$

and from (7.3.8)

$$c_3 = - CQ_2{}^4 - 4CQ_2 .$$

Thus

$$p = -C(Q^4+Q_2{}^4) + C(Q-Q_2) ,$$

and the mixed components of the convected stress tensor, which in this problem are identical to the physical components, are after being non-dimensionalized

$$\tau^1{}_1/k' = \frac{C}{k'} (Q^4-Q_2{}^4) + \frac{4C}{k'} (Q-Q_2) , \qquad (7.3.9)$$

and $$\tau^2{}_2/k' = \frac{2C}{k'} \frac{1}{Q^2} - \frac{C}{k'} (Q^4+Q_2{}^4) + \frac{4C}{k'} (Q-Q_2)$$

$$= \tau^3{}_3/k' . \qquad (7.3.10)$$

At the elastic-plastic boundary

$$\tau^2{}_2 - \tau^1{}_1 = Y ,$$

where $Y = 2k'$, $k'$ being the Tresca yield stress in pure shear. Thus from equations (7.3.9) and (7.3.10) the equation

$$Q_3{}^6 + \frac{k'}{C} Q_3{}^2 - 1 = 0 \qquad (7.3.11)$$

is obtained, where $Q_3 = Q(a)$. The value of $Q_3$ which must lie in the interval (0,1) may be found numerically from equation (7.3.11) for a given value of $k'/C$. Once $Q_3$ is known, the value of a is found from

equation (7.2.1) which at $r = a$ may be rearranged to give

$$a = \frac{R_1}{Q_1} \left\{ \frac{1-Q_1^{\,3}}{1-Q_3^{\,3}} \right\}^{1/3} .$$

The stresses in the elastic region and the radius of the elastic-plastic boundary can thus be found for any given state of deformation as specified by $Q_1$.

## 7.4 Elastic-Plastic Region

The yielded region of the shell in C.3 is given by

$$r_1 \leq r \leq a ,$$

and in this region the elastic strains are found using C.2 as the natural state. It has been noted previously that in general C.2 corresponds to a non-Euclidean space so that the components $G^{ij}$ cannot be found by a transformation of the metric tensor from Cartesian coordinates to the convected coordinates as was done for $G^{ij}$ and $g^{ij}$. Instead the neo-Hookean constitutive equation, the equilibrium equations, the incompressibility of the material, and the plastic yield condition determine the components of $G^{ij}$.

In the elastic-plastic region of the shell, $r_1 \leq r \leq a$, the stress is determined by the yield condition and the equilibrium equation.

Spherical symmetry gives that

$$\tau^{12} = \tau^{13} = \tau^{23} = 0 \ .$$

Thus the first equilibrium equation (Appendix C) becomes

$$\frac{d\tau^{11}}{dr} + \frac{2}{r}\tau^{11} - r\tau^{22} - r\sin^2\theta\tau^{33} = 0$$

or

$$\frac{d\tau^1{}_1}{dr} - \frac{2}{r}(\tau^2{}_2 - \tau^1{}_1) = 0 \ . \qquad (7.4.1)$$

The stresses must satisfy the yield condition

$$\tau^2{}_2 - \tau^1{}_1 = 2k'$$

so that equation (7.4.1) becomes

$$\frac{d\tau^1{}_1}{dr} = \frac{4k'}{r}$$

and after integration is

$$\tau^1{}_1 = 4k'\ln r + c_4$$

with

$$\tau^2{}_2 = \tau^3{}_3 = \tau^1{}_1 + 2k' \ .$$

The constant of integration $c_4$ is determined from the condition

that $\tau^1{}_1$ be continuous at $r = a$. The resulting stress solution is

$$\tau^1{}_1/k' = 4 \ln\left(\frac{r}{a}\right) + \frac{C}{k'}\left(Q_3{}^4 - Q_2{}^4\right) + \frac{4C}{k'}\left(Q_3 - Q_2\right) ,$$

$$\tau^2{}_2/k' = \tau^3{}_3/k' = \tau^1{}_1/k' + 2 .$$

The components of the metric tensor $G^{ij}$ in C.2 may now be found. In the yielded region the natural state is C.2 so that

$$\tau^{ij} = 2CG^{ij} + pg^{ij} \tag{7.4.2}$$

and therefore $G^{12}$, $G^{13}$, and $G^{23}$ are zero.

The condition that the deformation from C.2 to C.3 occur without volume change gives

$$g/G = 1$$

where $g = \det(g_{ij})$ and $G = \det(G_{ij})$ so that

$$G^{11}G^{22}G^{33}\; r^4 \sin^2\theta = 1 . \tag{7.4.3}$$

Substitution of equation (7.4.2) into the second equilibrium equation (7.3.5) gives

$$\frac{1}{r^2}\frac{\partial p}{\partial \theta} + 2CG^{22}\cot\theta - 2CG^{33}\sin\theta\cos\theta = 0$$

and since $\partial p/\partial\theta$ is zero from symmetry

$$G^{33} = G^{22}/\sin^2\theta. \tag{7.4.4}$$

Thus from equations (7.4.3) and (7.4.4) it follows that

$$G^{22} = \frac{1}{r^2\sqrt{G^{11}}} . \tag{7.4.5}$$

The yield condition and equations (7.4.2), (7.4.4), and (7.4.5) give

$$2C(\frac{1}{\sqrt{G^{11}}} - G^{11}) = 2k'$$

or

$$(G^{11})^{3/2} + \frac{k'}{C}(G^{11})^{1/2} - 1 = 0 . \tag{7.4.6}$$

This implies that $G^{11}$ is constant throughout the yielded region and comparison of equations (7.3.11) and (7.4.6) shows that

$$G^{11} = Q_3{}^4 . \tag{7.4.7}$$

The constant value of $G^{11}$ determined from equation (7.4.7) is henceforth denoted by $\gamma$.

A complete solution for the stresses in the shell has thus been found for any given state of deformation in C.3. It remains now to determine the residual stresses which result from removal of the

internal pressure.

## 2.5 Residual Stresses

It is convenient to introduce another convected coordinate system defined in a manner similar to the definition of the coordinates $\xi_i$. Define the coordinates $\xi_i'$ by

$$\xi_1' = r,$$

$$\xi_2' = \theta,$$

and

$$\xi_3' = \phi,$$

where $(r,\theta,\phi)$ are the coordinates of a particle in C.4 referred to the spherical polar coordinate system. Furthermore let $G'^{ij}$, $\mathcal{G}'^{ij}$, and $g'^{ij}$ be the contravariant components of the metric tensor referred to the convected coordinates $\xi_i'$ in C.1, C.2, and C.4 respectively, and denote by $t^{ij}$ the contravariant components of the stress tensor in C.4 referred to this convected coordinate system.

Since the components of the metric tensor in C.2 referred to the $\xi_i$ coordinate system are known, the components $\mathcal{G}'^{ij}$ may be found from

$$\mathcal{G}'^{ij} = \frac{\partial \xi'^i}{\partial \xi^m} \frac{\partial \xi'^j}{\partial \xi^n} \mathcal{G}^{mn}$$

and since

$$\left[\frac{\partial \xi'^i}{\partial \xi^j}\right] = \begin{bmatrix} \frac{d\mathcal{r}}{dr} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if follows that

$$[G'^{ij}] = \begin{bmatrix} \left(\frac{d\mathcal{r}}{dr}\right)^2 \gamma & 0 & 0 \\ 0 & \frac{1}{r^2\sqrt{\gamma}} & 0 \\ 0 & 0 & \frac{1}{r^2\sqrt{\gamma}\sin^2\theta} \end{bmatrix} .$$

The components of $g'^{ij}$ are

$$[g'^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\mathcal{r}^2} & 0 \\ 0 & 0 & \frac{1}{\mathcal{r}^2\sin^2\theta} \end{bmatrix}$$

The elastic deformation between C.2 and C.4 is isochoric so that

$$g'/G' = 1 ,$$

where $$g' = \det(g'_{ij}) \text{ and } G' = \det(G'_{ij})$$

and therefore

$$\frac{d\textit{r}}{dr} = \left(\frac{r}{\textit{r}}\right)^2 . \tag{7.5.1}$$

Letting $T = \textit{r}/r$, the condition of incompressibility gives

$$T(\textit{r}) = \left\{\frac{1}{1 - \left(\frac{a^3 - \mathrm{a}^3}{\textit{r}^3}\right)}\right\}^{1/3} . \tag{7.5.2}$$

Thus the components of $G'^{ij}$ are, after use of equations (7.4.6) and (7.5.1)

$$[G'^{ij}] = \begin{bmatrix} \frac{\gamma}{T^4} & 0 & 0 \\ 0 & \frac{T^2}{\textit{r}^2}\left(\gamma + \frac{k'}{C}\right) & 0 \\ 0 & 0 & \frac{T^2\left(\gamma + \frac{k'}{C}\right)}{\textit{r}^2 \sin^2\theta} \end{bmatrix} .$$

In the region of the shell in C.4 which consists of the material which yielded in C.3, the elastic strains are referred to the natural state C.2 so that

$$t^{ij} = 2CG'^{ij} + pg'^{ij} ,$$

and therefore

$$t^{11} = 2C \frac{\gamma}{T^4} + p \tag{7.5.3}$$

and

$$t^{22} = 2C \frac{T^2}{r^2} (\gamma + \frac{k'}{C}) + \frac{p}{r^2} = \sin^2\theta t^{33} \ . \tag{7.5.4}$$

The equilibrium equations (7.3.5) and (7.3.6), referred to the $\xi_i'$ coordinates are satisfied identically due to symmetry, and the first equilibrium equation gives

$$2C\gamma \frac{d}{dr} (\frac{1}{T^4}) + \frac{dp}{dr} - \frac{4C}{r} T^2(\gamma + \frac{k'}{C}) + \frac{4C\gamma}{rT^4} = 0 \ . \tag{7.5.5}$$

From the definition of T and equation (7.5.1)

$$\frac{d}{dr} (\frac{1}{T^4}) = - \frac{4}{T^5} \frac{dT}{dr} = \frac{4(T^3-1)}{rT^4}$$

so that from equation (7.5.5)

$$\frac{dp}{dr} = \frac{4C\gamma}{r} \frac{(T^3-1)^2}{T^4} + \frac{4k'T^2}{r}$$

and thus

$$\frac{dp}{dT} = 4C\gamma (\frac{1-T^3}{T^5}) + \frac{k'T}{1-T^3} \ .$$

Integration gives

$$p = C\gamma(\frac{4T^3-1}{T^4}) + \frac{2k'}{3}\ln[\frac{1+T+T^2}{(T-1)^2}]$$

$$+ \frac{4k'}{\sqrt{3}}\tan^{-1}(\frac{2T+1}{\sqrt{3}}) + c_5 . \tag{7.5.6}$$

The radial stress at the inner surface of the sphere in C.4 is zero so that from equations (7.5.3) and (7.5.6) the integration constant $c_5$ may be determined. This results in the following stress solution for the region $r_1 \leq r \leq a$,

$$t^1{}_1/k' = \frac{C\gamma}{k'}(\frac{1}{T^4} - \frac{1}{T_1{}^4}) + \frac{4C\gamma}{k'}(\frac{1}{T} - \frac{1}{T_1})$$

$$+ \frac{2}{3}\ln[\frac{(1+T+T^2)(T_1-1)^2}{(1+T_1+T_1{}^2)(T-1)^2}]$$

$$+ \frac{4}{\sqrt{3}}[\tan^{-1}(\frac{2T+1}{-\sqrt{3}}) - \tan^{-1}(\frac{2T_1+1}{-\sqrt{3}})] \tag{7.5.7}$$

and

$$t^2{}_2/k' = t^3{}_3/k' = t^1{}_1/k' - \frac{2C\gamma}{k'T^4} + \frac{2CT^2}{k'}(\gamma + \frac{k'}{C}) , \tag{7.5.8}$$

where $T_1 = T(r_1)$ .

The value of a is known from the solution in C.3 and $a$, which appears in T, is to be determined from the continuity of $t^{11}$ at $r = a$.

In the region $a \leq r \leq r_2$ the elastic strains in C.4 are referred to the natural state C.1 so that the neo-Hookean constitutive equation in this region is

$$t^{ij} = 2CG'^{ij} + pg'^{ij} .$$

The components of $G'^{ij}$, in analogy to the components of $G^{ij}$ are found to be

$$[G'^{ij}] = \begin{bmatrix} S^4 & 0 & 0 \\ 0 & \frac{1}{r^2 S} & 0 \\ 0 & 0 & \frac{1}{r^2 S^2 \sin^2\theta} \end{bmatrix} ,$$

where $S = R/r$ . From the incompressibility of the material it follows that

$$S = \left\{ 1 + \frac{A^3 - a^3}{r^3} \right\}^{1/3} .$$

The stresses are found in a manner similar to that used to obtain the stresses in the elastic region of C.3. This gives

$$t^1{}_1/k' = \frac{C}{k'} (S^4 - S_2{}^4) + \frac{4C}{k'} (S - S_2) \qquad (7.5.9)$$

and $$t^2{}_2/k' = t^3{}_3/k' = \frac{2C}{k'S^2} - \frac{C}{k'}(S^4+S_2{}^4) + \frac{4C}{k'}(S-S_2) \; , \qquad (7.5.10)$$

where $S_2 = S(r_2)$ .

If the value of $a$ is known, equations (7.5.7), (7.5.8), (7.5.9), and (7.5.10) specify completely the stress solution. Continuity of $t^{11}$ at $r = a$ gives

$$\gamma(\frac{1}{T_3{}^4} - \frac{1}{T_1{}^4}) + 4\gamma(\frac{1}{T_3} - \frac{1}{T_1})$$

$$+ \frac{2k'}{3C} \ln [\frac{(1+T_3+T_3{}^2)(T_1-1)^2}{(1+T_1+T_1{}^2)(T_3-1)^2}] + \frac{4k'}{\sqrt{3}C} [\tan^{-1}(\frac{2T_3+1}{-\sqrt{3}}) - \tan^{-1}(\frac{2T_1+1}{-\sqrt{3}})]$$

$$- (S_3{}^4-S_2{}^4) - 4(S_3-S_2) = 0 \; , \qquad (7.5.11)$$

where $T_3 = T(a)$ and $S_3 = S(a)$. The only unknown in equation (7.5.11) is $a$ and it may be found using a numerical procedure. The computer program which was written to evaluate the numerical results for this problem uses an interval halving technique [72] to solve equation (7.5.11) for $a$.

## 7.6 Possibility of Further Yielding on Unloading

It is found that the assumption that yielding does not occur during unloading from C.3 is not always valid. The alterations required in the residual stress solution when yielding does occur are discussed in this section.

If it is found using the analysis of section 7.5 that the yield condition is violated by the residual stress solution then the solution in the region $r_1 \leq r \leq a$ must be considered in two parts. In C.4 the region in which further yielding occurs upon unloading is assumed to be given by

$$r_1 \leq r \leq b \ .$$

The stress solution in the region

$$b \leq r \leq a$$

is given by equations (7.5.3), (7.5.4), and (7.5.6) but the integration constant $c_5$ now has a different value which is yet to be determined.

At $r = b$ the stresses satisfy the yield condition

$$t^1{}_1 - t^2{}_2 = 2k' \ , \tag{7.6.1}$$

which gives

$$(1 + \frac{k'}{C\gamma})\ T_4{}^6 + \frac{k'}{C\gamma}\ T_4{}^4 - 1 = 0 \ , \tag{7.6.2}$$

where $T_4 = T(b)$. The value of $T_4$ may be found from equation (7.6.2) using a numerical procedure.

Using equation (7.5.2) the following equation is obtained which determines the value of $b$.

$$b = a\,T_4\left\{\frac{1-T_3{}^3}{1-T_4{}^3}\right\}^{1/3} .$$

In the yielded region, $r_1 \leq r \leq b$, using the yield condition (7.6.1) and following an analysis similar to that used to obtain the stresses in the elastic-plastic region of C.3 gives

$$t^1{}_1/k' = -4\ln\left(\frac{r}{r_1}\right) ,$$

and
$$t^2{}_2/k' = t^3{}_3/k' = -4\ln\left(\frac{r}{r_1}\right) - 2 .$$

The new value of the integration constant $c_5$ in equation (7.5.6) is found from continuity of $t^{11}$ at $r = b$ so that in the region $b \leq r \leq a$ the stress solution is

$$t^1{}_1/k' = \frac{C\gamma}{k'}\left(\frac{1}{T^4} - \frac{1}{T_4{}^4}\right) + \frac{4C\gamma}{k'}\left(\frac{1}{T} - \frac{1}{T_4}\right)$$

$$+ \frac{2}{3}\ln\left[\frac{(1+T+T^2)(T_4-1)^2}{(1+T_4+T_4{}^2)(T-1)^2}\right] + \frac{4}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2T+1}{-\sqrt{3}}\right) - \tan^{-1}\left(\frac{2T_4+1}{-\sqrt{3}}\right)\right.$$

$$\left. + 4\ln\left(\frac{r}{r_1}\right)\right]$$

and
$$t^2{}_2/k' = t^3{}_3/k' = t^1{}_1/k' + \frac{2CT^2}{k'}\left(\gamma + \frac{k'}{C}\right) - \frac{2C}{k'}\frac{\gamma}{T^4} .$$

The condition that the radial stress be continuous at $r = a$ gives

$$\frac{C\gamma}{k'}\left(\frac{1}{T_3^{\,4}} - \frac{1}{T_4^{\,4}}\right) + \frac{4C\gamma}{k'}\left(\frac{1}{T_3} - \frac{1}{T_4}\right)$$

$$+ \frac{2}{3}\ln\left[\frac{(1+T_3+T_3^{\,2})(T_4-1)^2}{(1+T_4+T_4^{\,2})(T_3-1)^2}\right] + \frac{4}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2T_3+1}{-\sqrt{3}}\right) - \tan^{-1}\left(\frac{2T_4+1}{-\sqrt{3}}\right)\right]$$

$$+ 4\ln\left(\frac{a}{r_1}\right) - (S_3^{\,4}-S_2^{\,4}) - 4(S_3-S_2) = 0 \qquad (7.6.3)$$

in which $a$ is the unknown. Once $a$ is found from equation (7.6.3) the complete residual stress solution is known.

7.7 Discussion

A computer program was written to evaluate numerical results for the stress solutions obtained in this chapter. These results are shown in Figures 7.2 to 7.18.

The stress solutions in C.3 for various values of $Q_1$ and the associated residual stress solutions are given for shells with $R_2/R_1$ equal to 2 and 10, with values of k'/C equal to 0.1, 0.5, and 1.0.

Figures 7.14 and 7.15 which give P/k' versus $u/R_1$ where P and u are the internal pressure and the displacement of the inner surface respectively. All the shells considered are seen to become unstable before they become fully plastic. That is, a point is reached during loading at which the internal pressure required to maintain a

given deflection decreases with increasing deflection. For example a shell with $R_2/R_1 = 10$ and $k'/C = 1.0$ does not become fully plastic until $u/R_1$ equals 8.186 but becomes unstable at about $u/R_1 = 1.8$. Thus the large elastic strains associated with values of $k'/C$ of the order of 0.1 to 1.0 result in significant changes in the surface area of the shells which have a considerable weakening effect.

The expansion of a shell beyond the elastic limit results in a strengthening of the shell after unloading in that a higher internal pressure is required to cause yielding in the shell upon renewed loading because of the residual stresses in C.4. For the classical elastic-plastic problem, Hill [70] has noted that a shell may not be strengthened by more than a factor of two. That is $P_*/P_o \leq 2$ where $P_*$ is the internal pressure in the state C.3 for which the material at the inner surface will be just on the point of yielding if the internal pressure is removed, and $P_o$ is the internal pressure at which yielding begins during the first loading of the shell. In the small strain theory where superposition of stress solutions is possible, $P_*$ is also the pressure at which yielding occurs during a second loading of the shell. Reuss [11] has shown that this maximum strengthening can be obtained only if $R_2/R_1 \geq 1.701$.

The solution found in this chapter reduces to that from the small strain theory. Using $k'/C$ equal to 0.001 which is of the same order of magnitude as that for most metals, the solution found here gives $P_*/P_o$ equal to 2.00 and a critical shell thickness given by

1.81. For shells with $R_2/R_1$ of the order of 1 and with $k'/C << 1$ the values of $\alpha$ and a are almost identical and a small numerical error in the value of $\alpha$ is likely to produce significant errors in the residual stress solution. It is believed that this is the reason for the small discrepancy in the critical shell thickness when $k'/C = 0.001$.

In Figures 7.16 to 7.18, for shells with $k'/C$ equal to 0.1, 0.5, and 1.0 respectively, curves are drawn which give $u_o/R_1$ and $u_*/R_1$ as functions of $R_2/R_1$ where $u_o$ is the inner displacement in C.3 at which yielding first occurs and $u_*$ is the inner displacement in the state C.3 at which yielding first occurs upon unloading. The ratio $P_*/P_o$ is also plotted as a function of $R_2/R_1$.

It is seen that unlike in the small strain theory $P_*/P_o$ is not constant but decreases with increasing $k'/C$ and for a given $k'/C$ decreases with increasing $R_2/R_1$, reaching a constant value as $R_2/R_1$ becomes of the order of 10 or more. Similarly the value of $R_2/R_1$ below which no yielding can occur upon unloading is also dependent on $k'/C$ becoming larger as $k'/C$ increases.

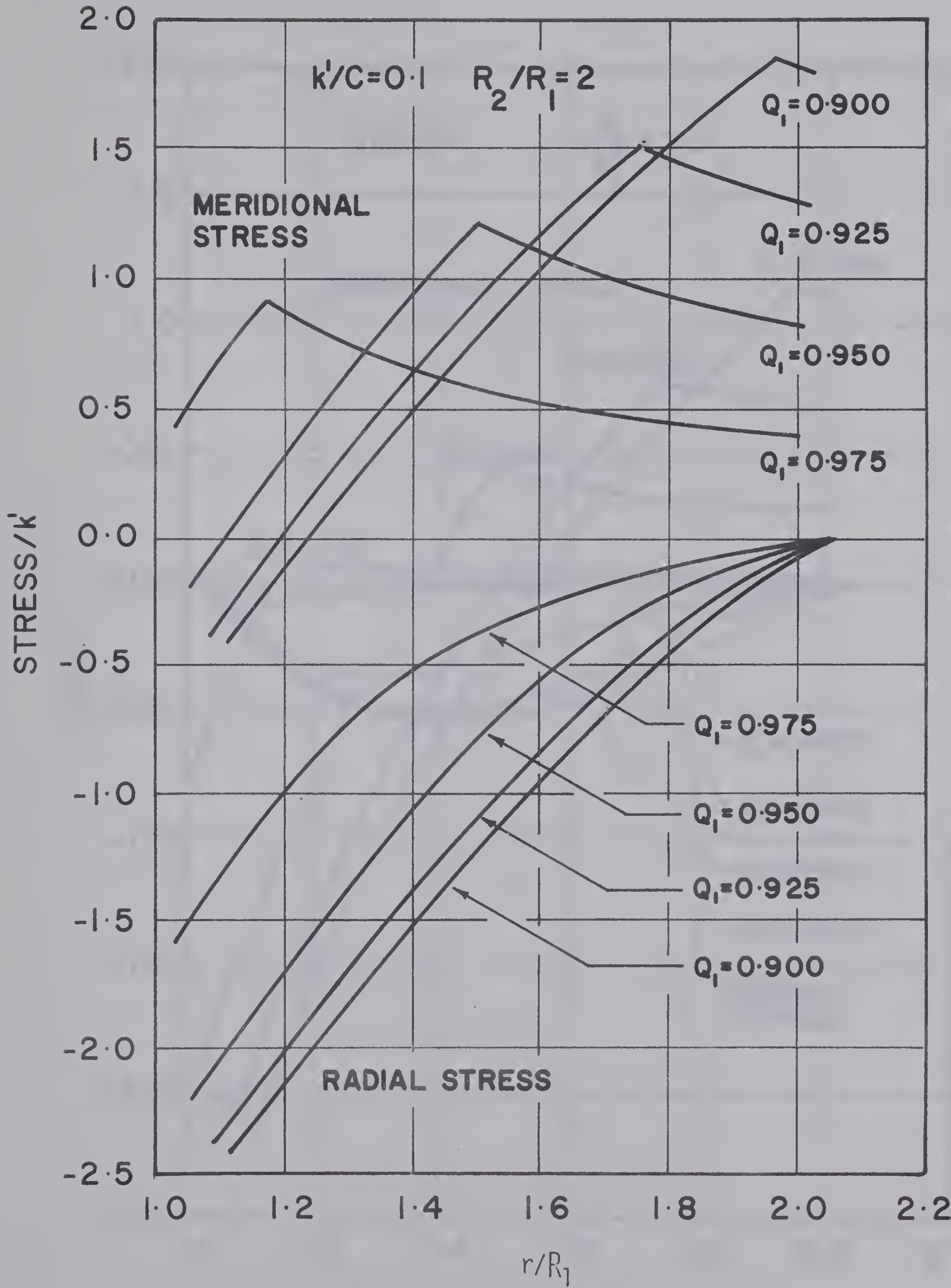
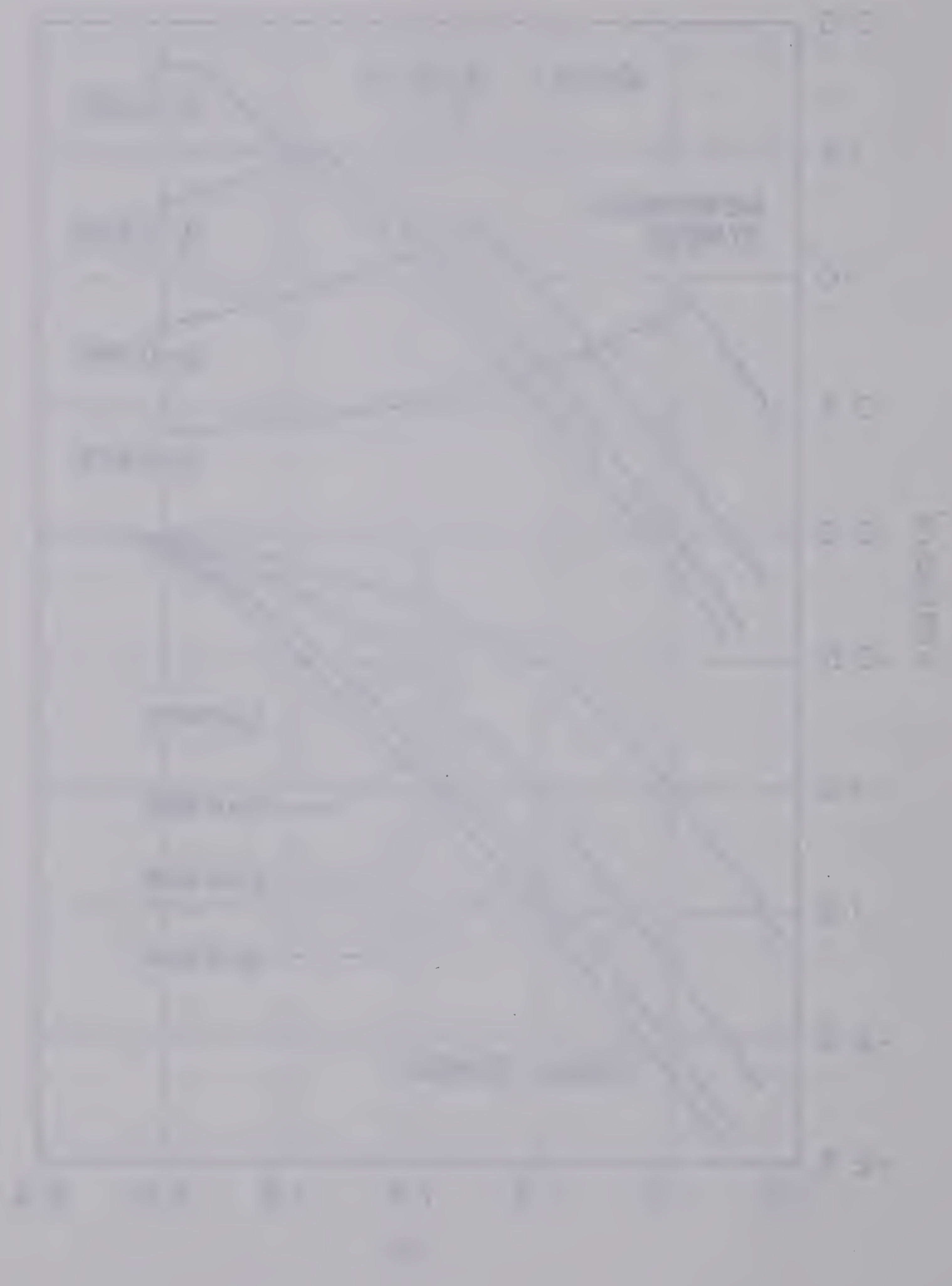



Figure 7.2 Stress Solution in C.3



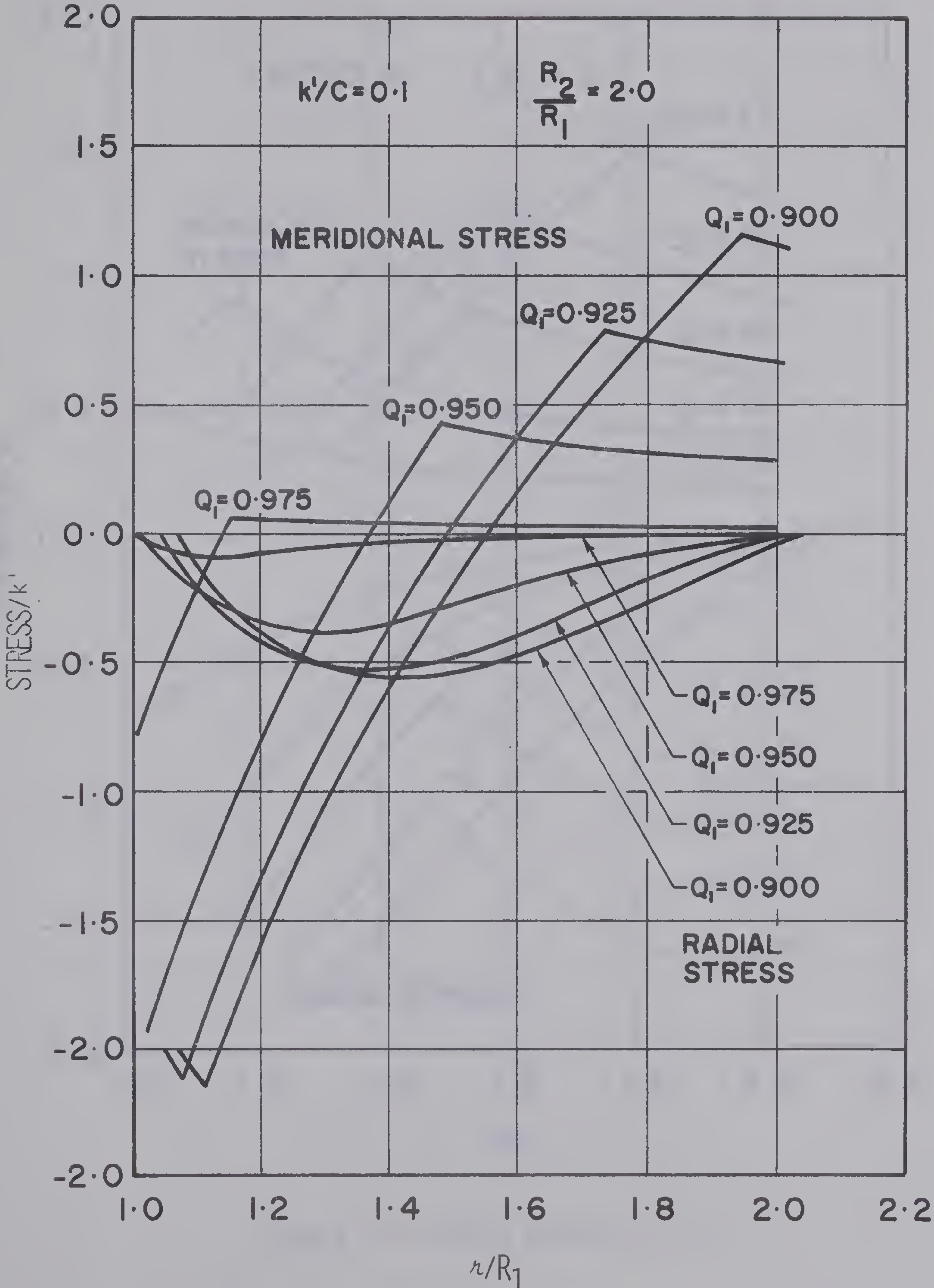



Figure 7.3 Residual Stresses

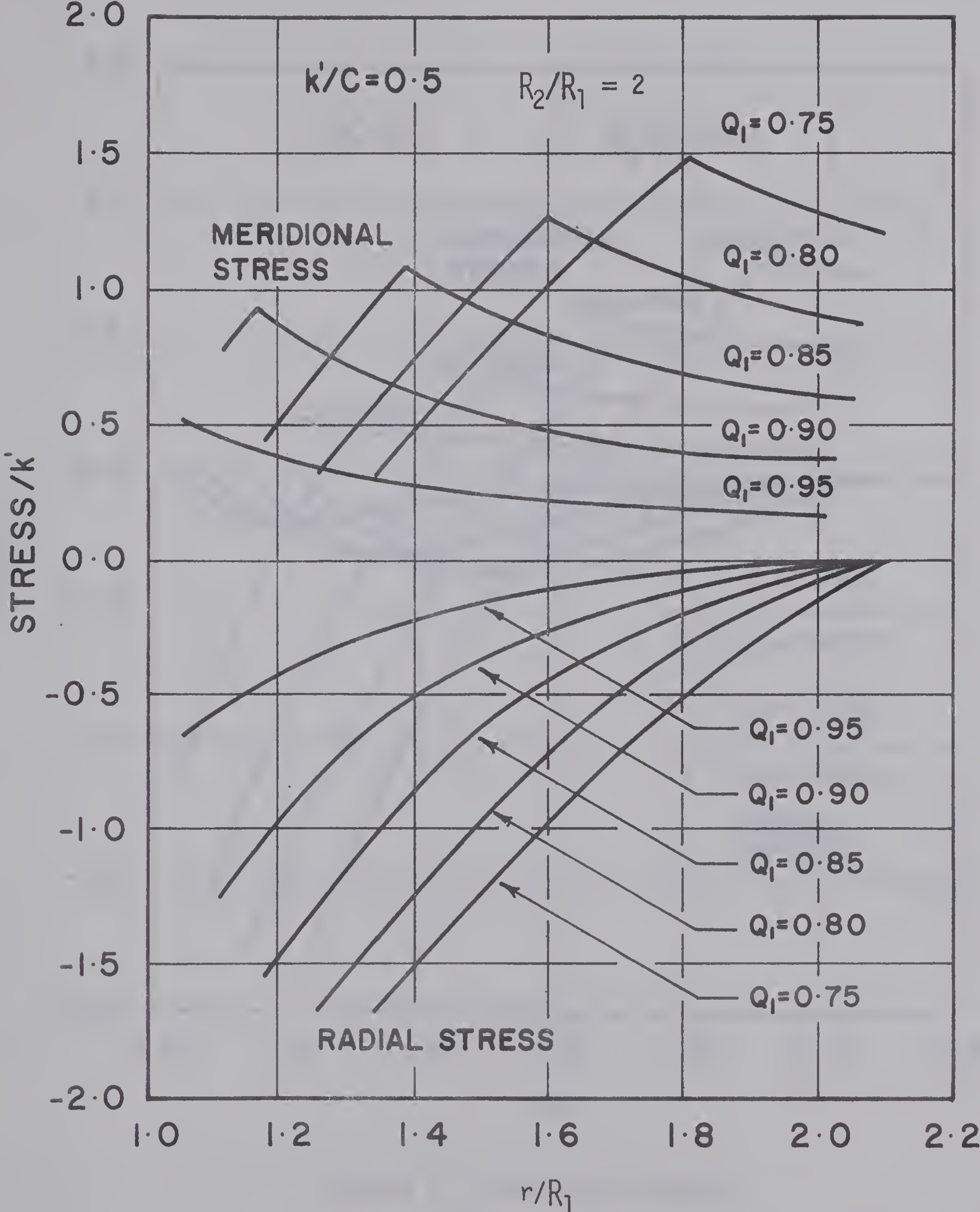



Figure 7.4 Stress Solution in C.3

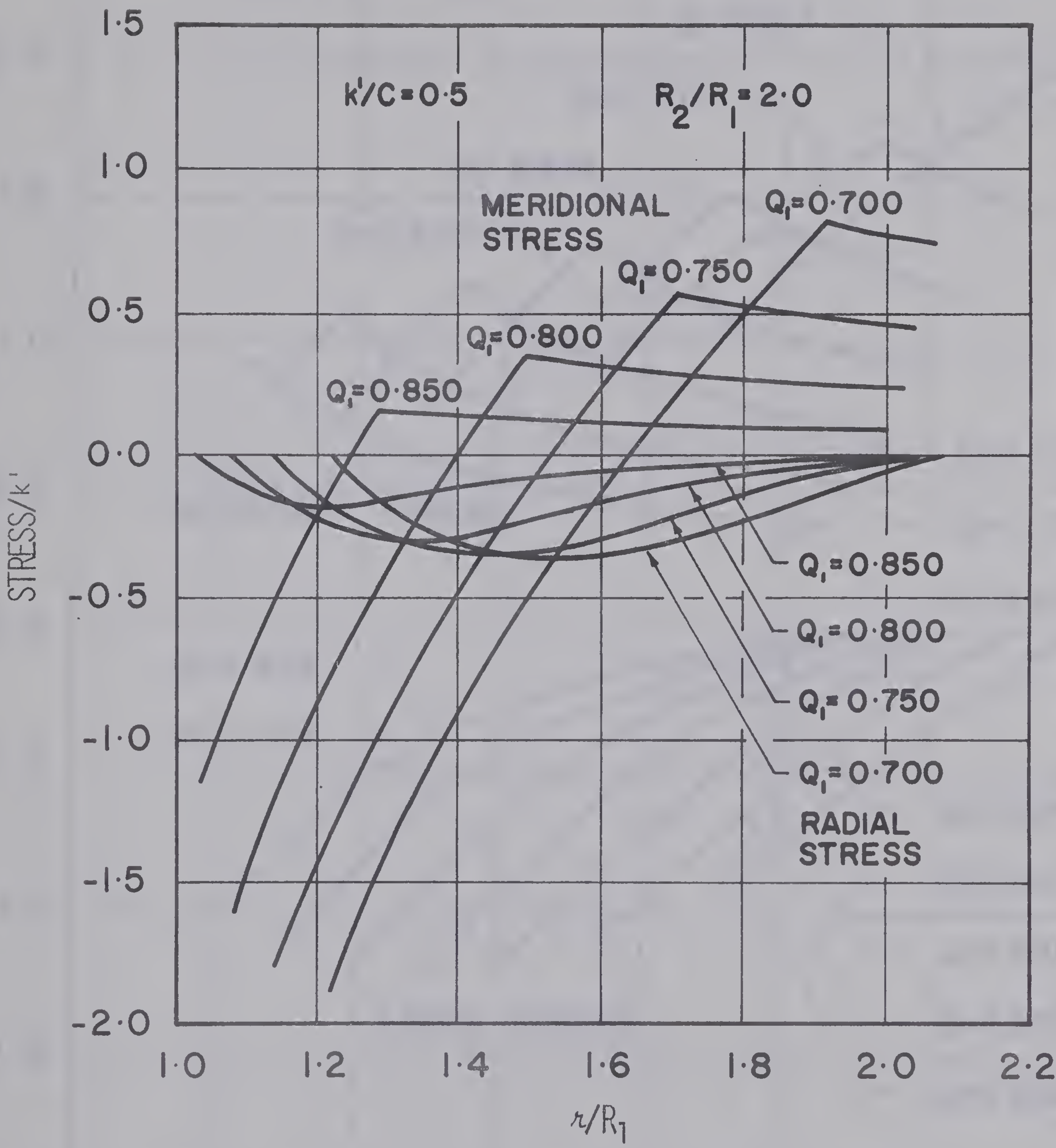
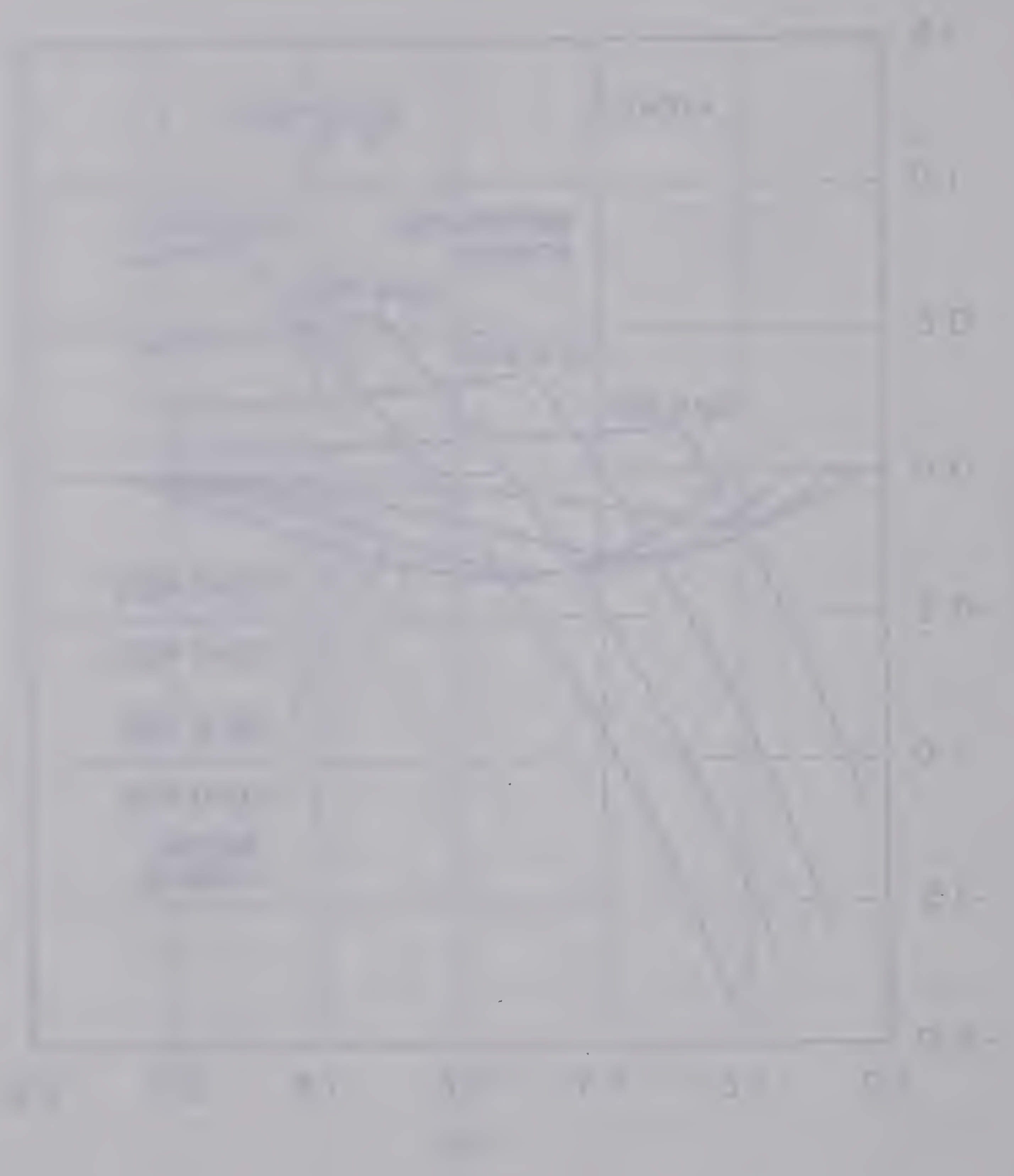



Figure 7.5 Residual Stresses



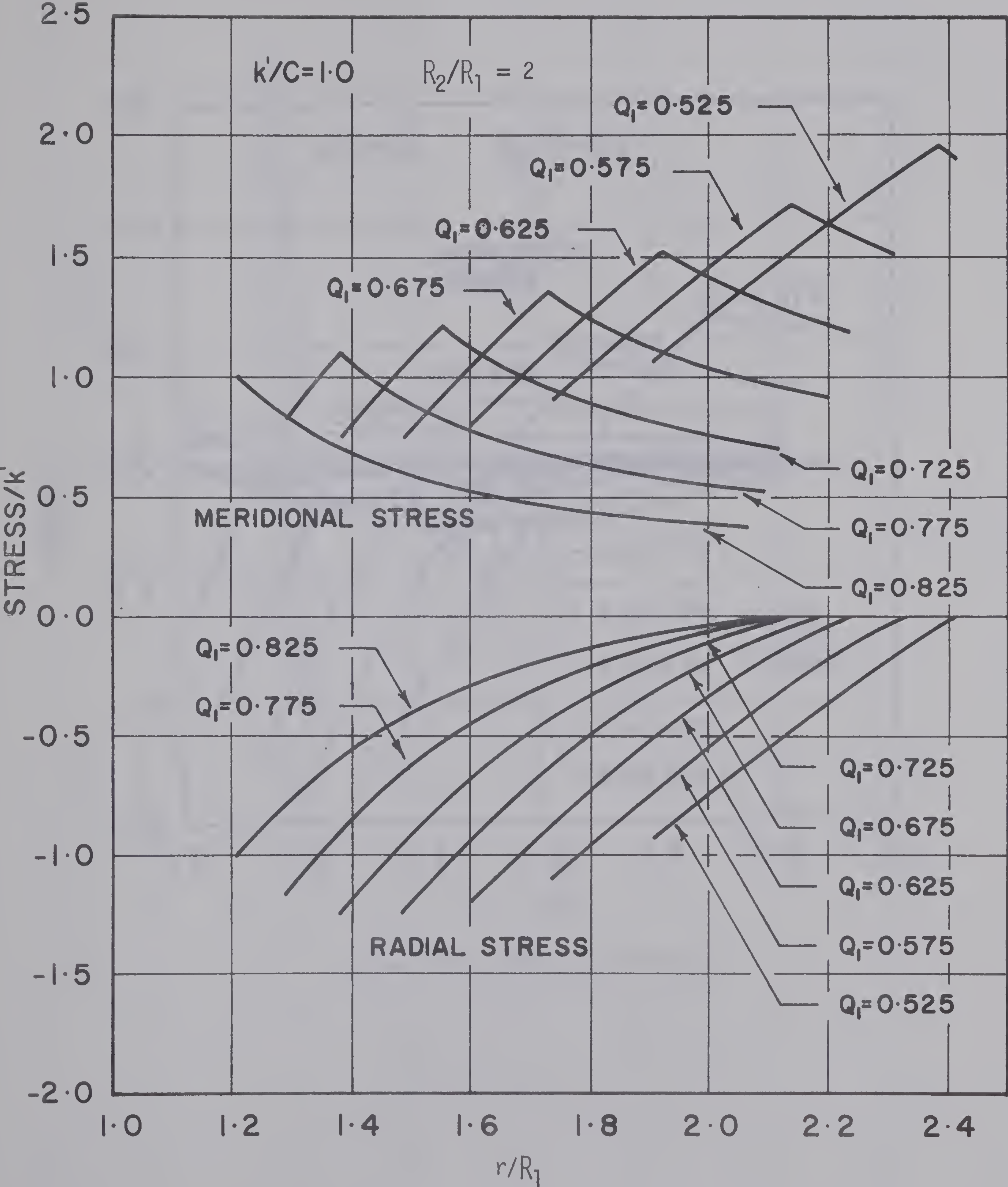
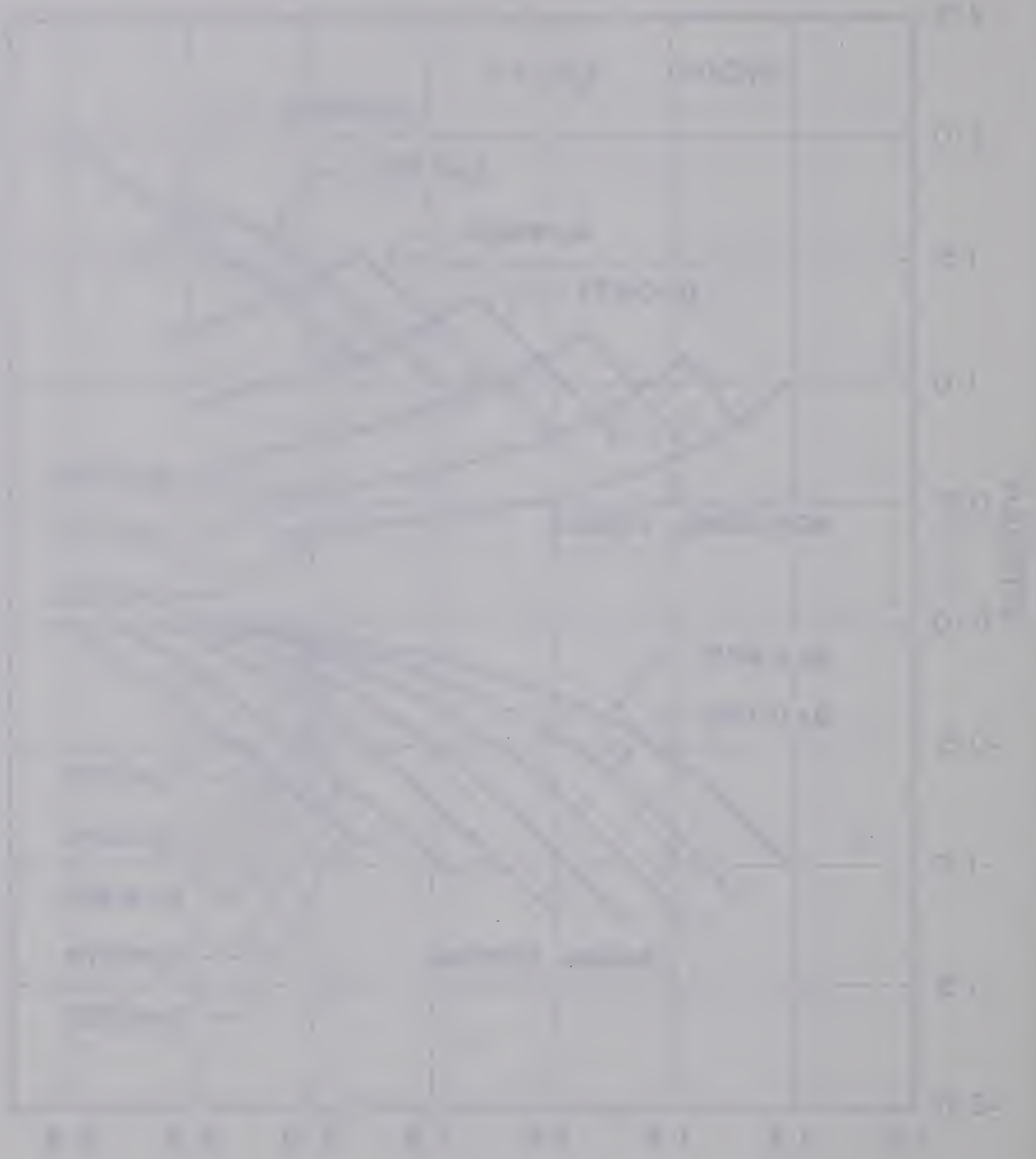



Figure 7.6 Stress Solution in C.3



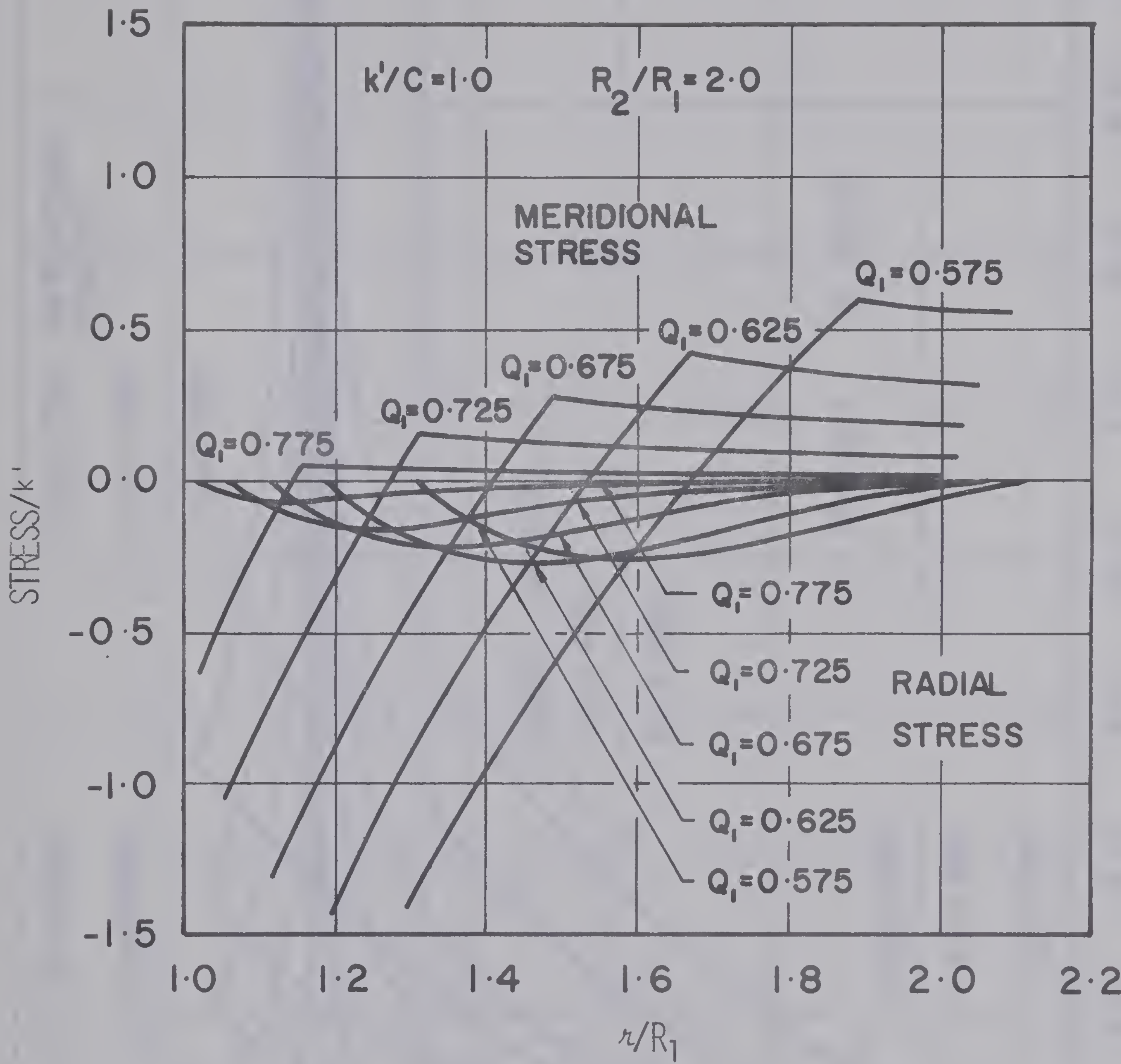



Figure 7.7 Residual Stresses

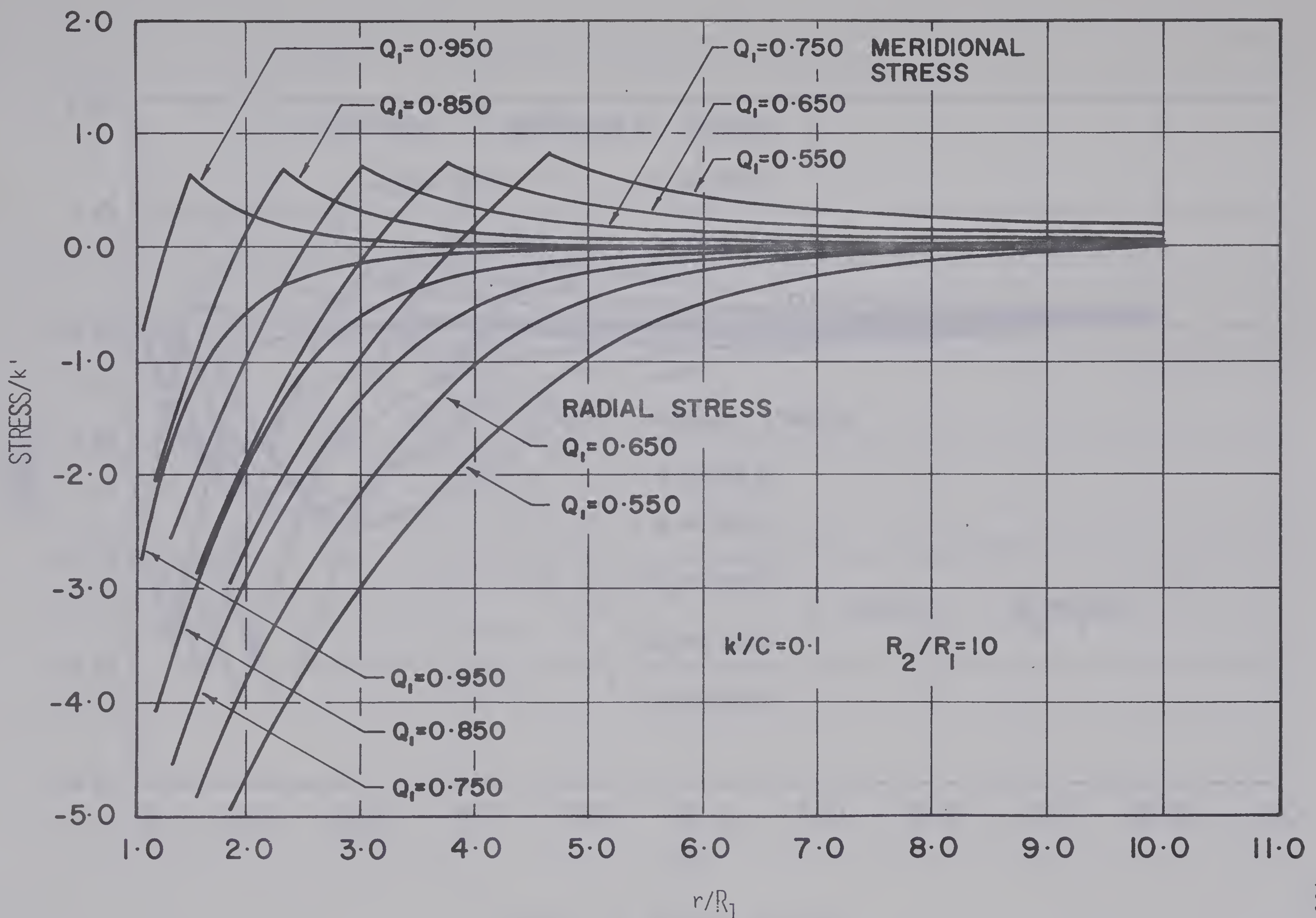
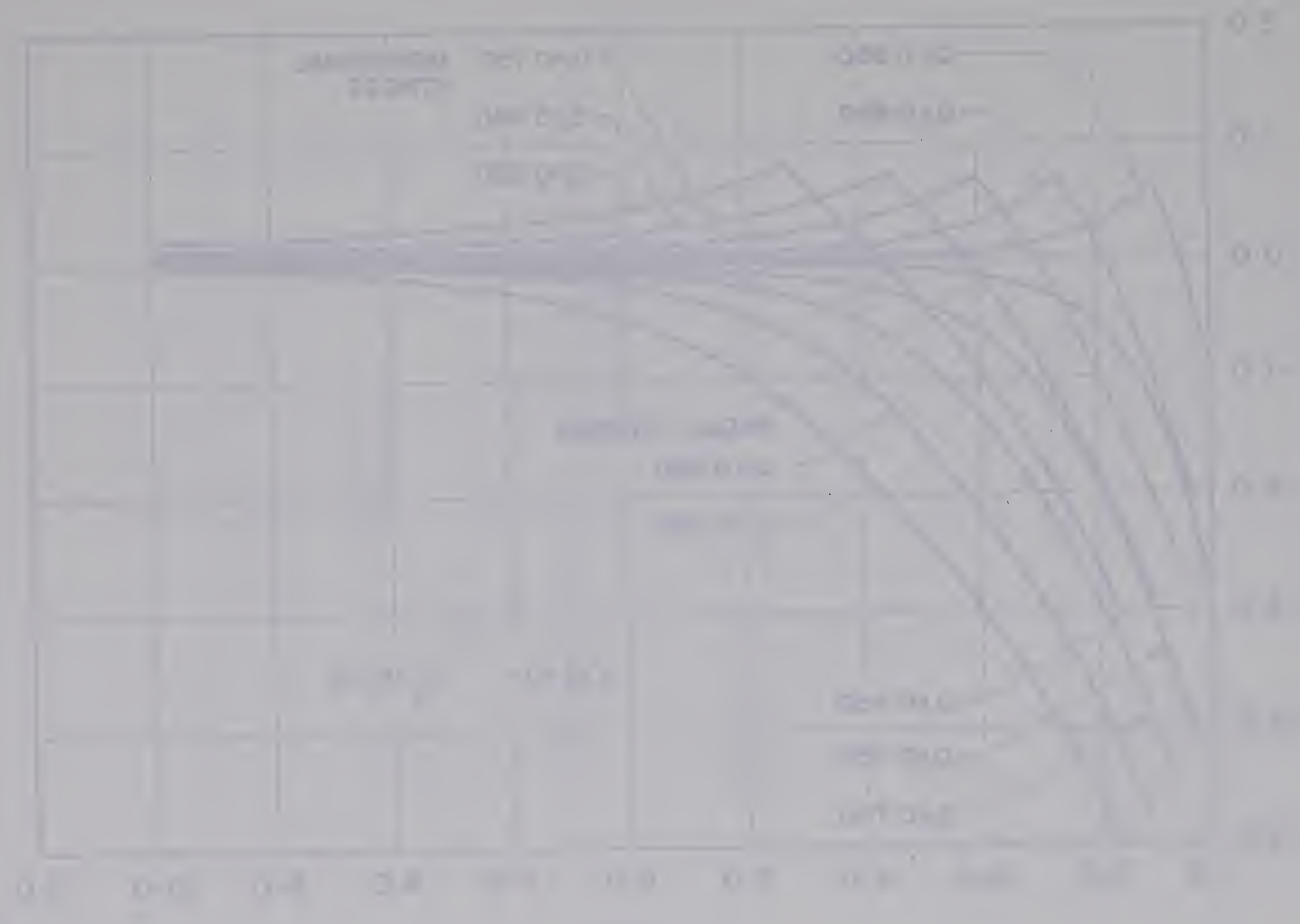



Figure 7.8 Stress Solution in C.3



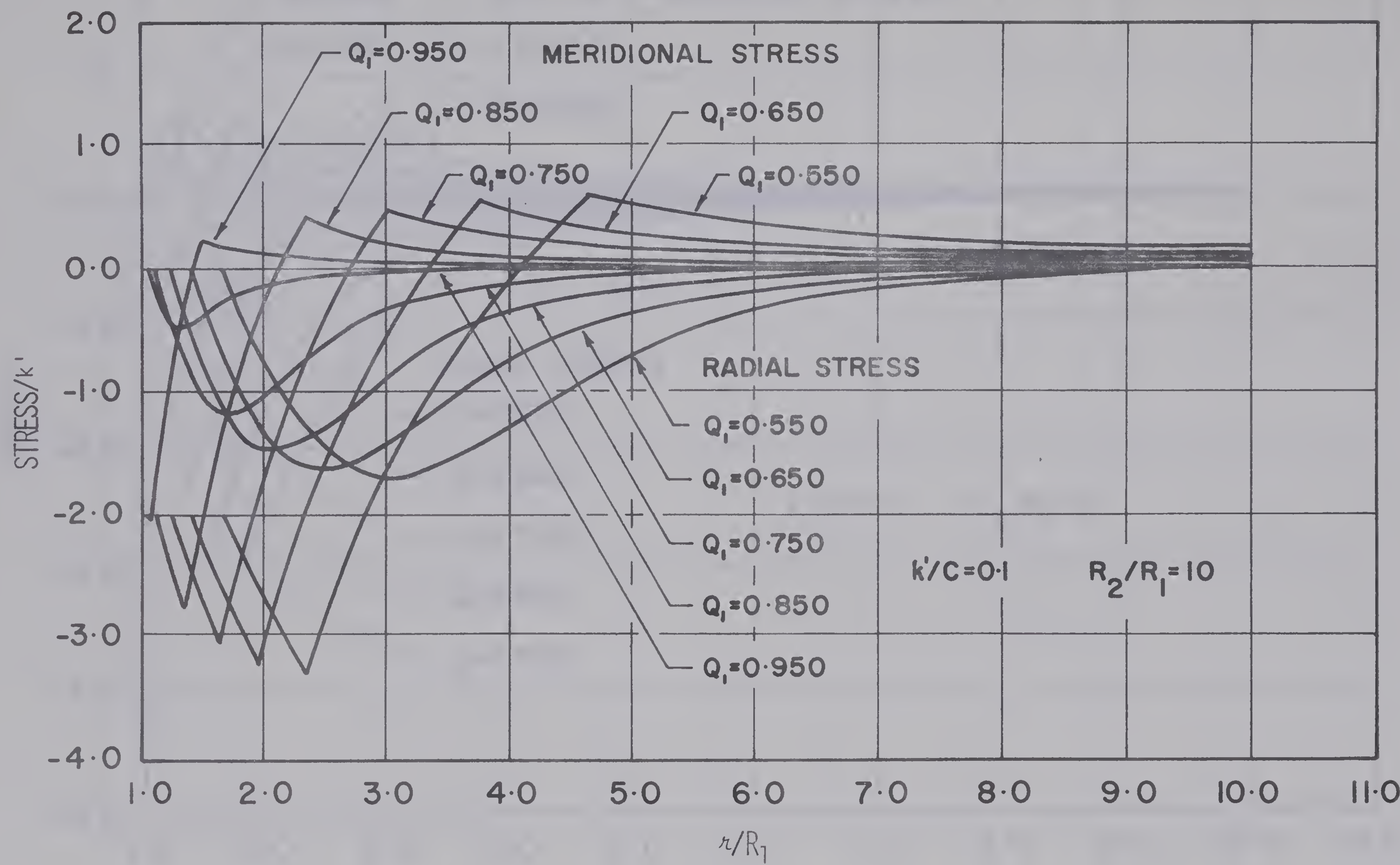
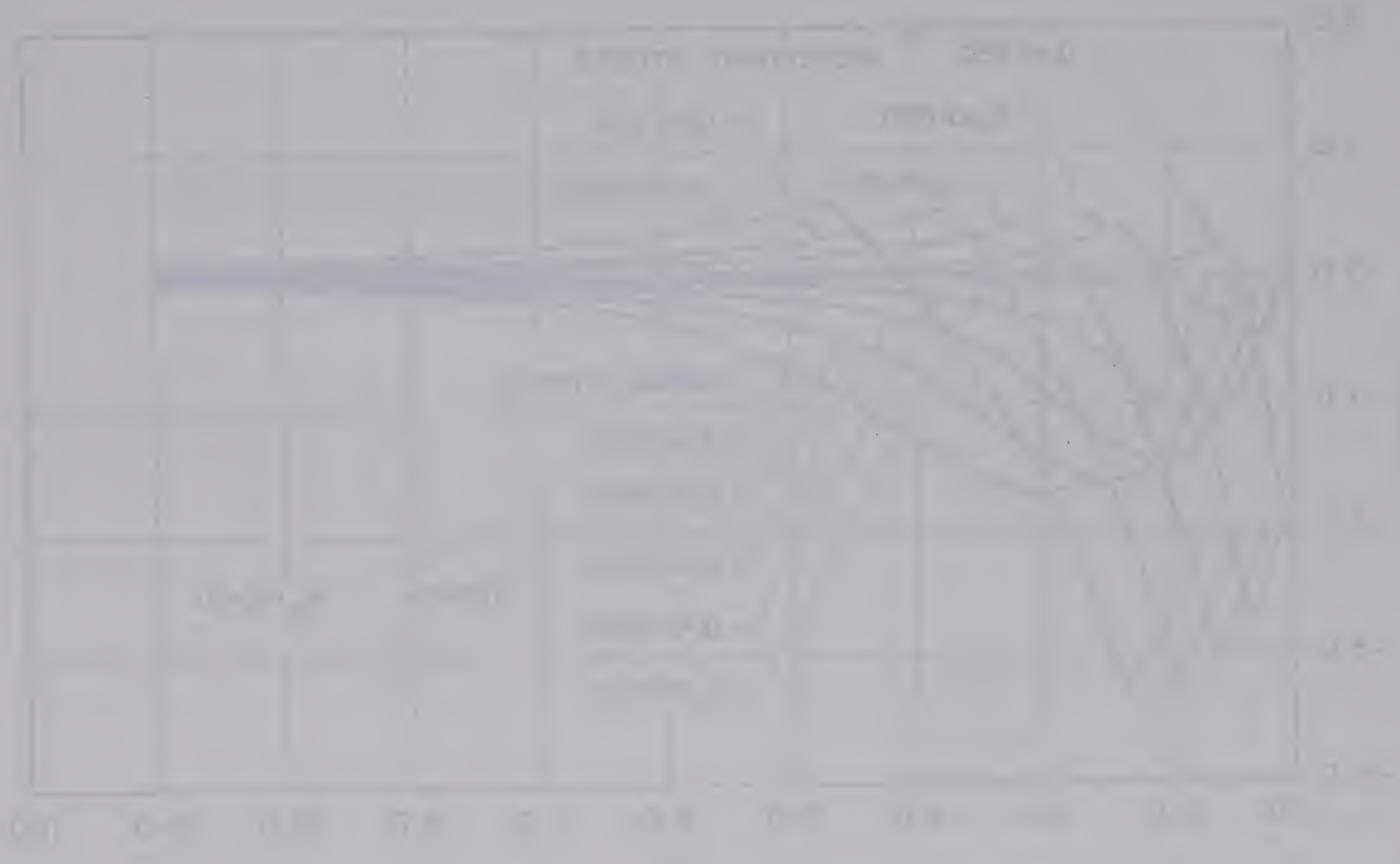



Figure 7.9 Residual Stresses



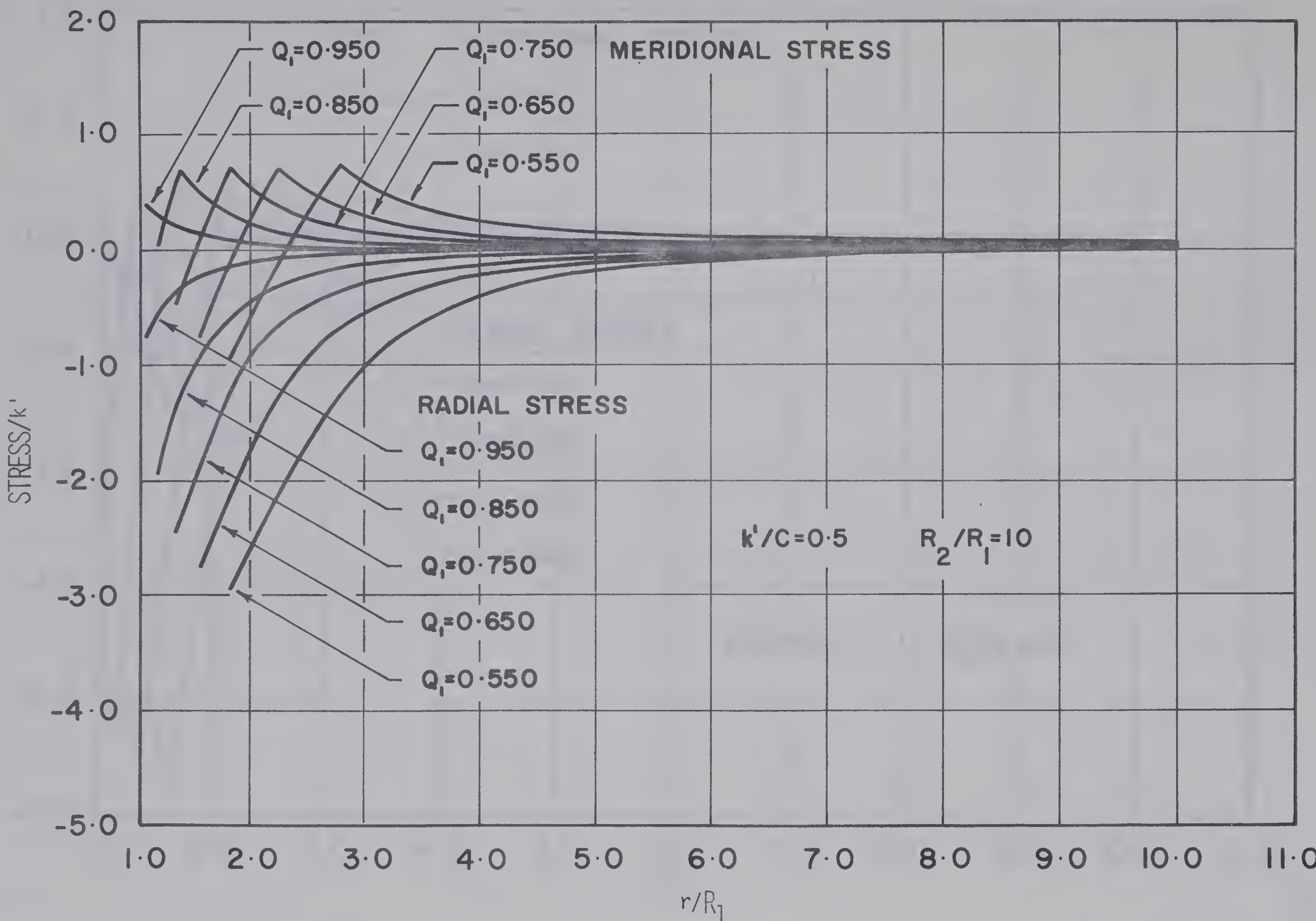
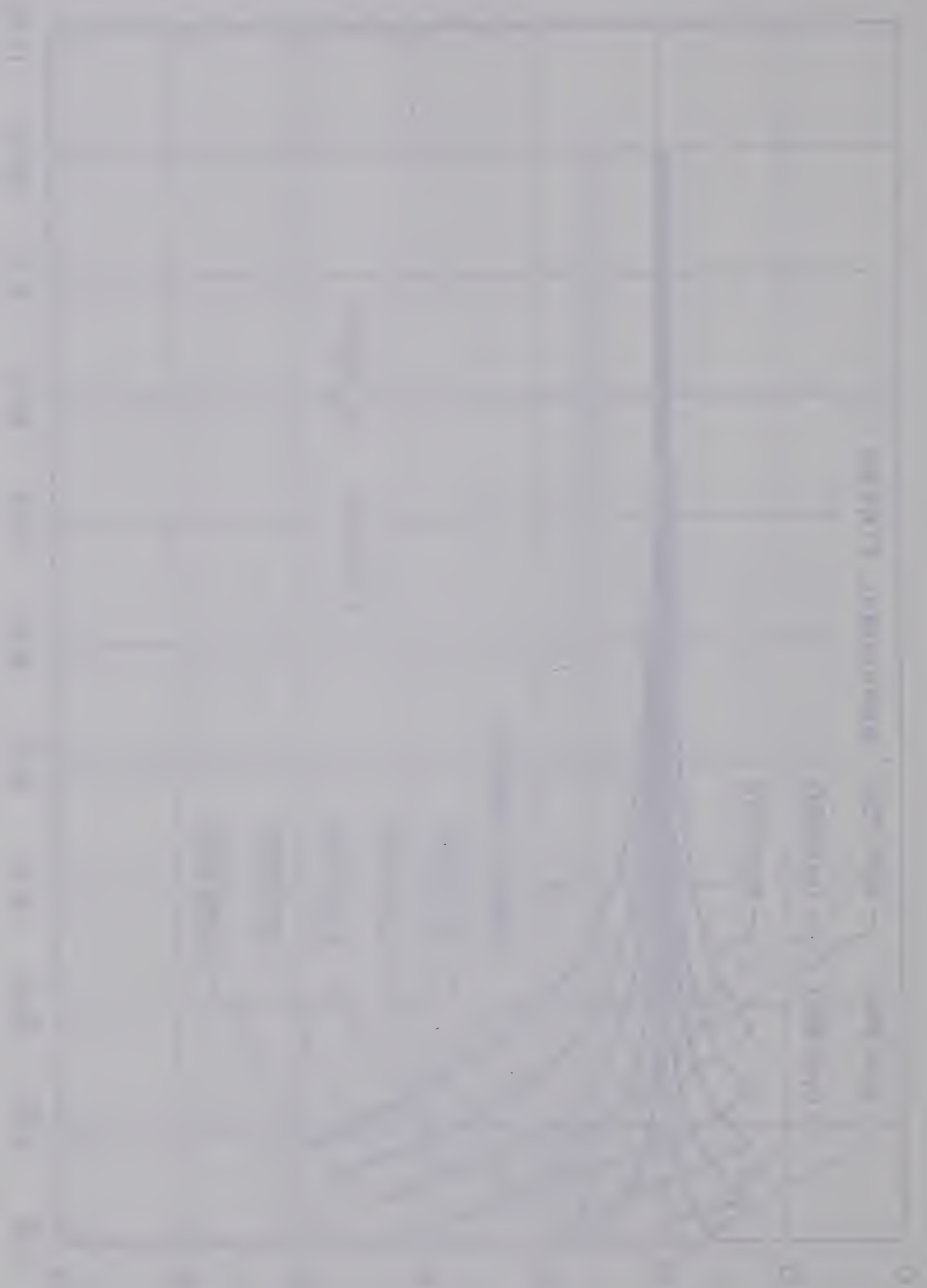



Figure 7.10 Stress Solution in C.3



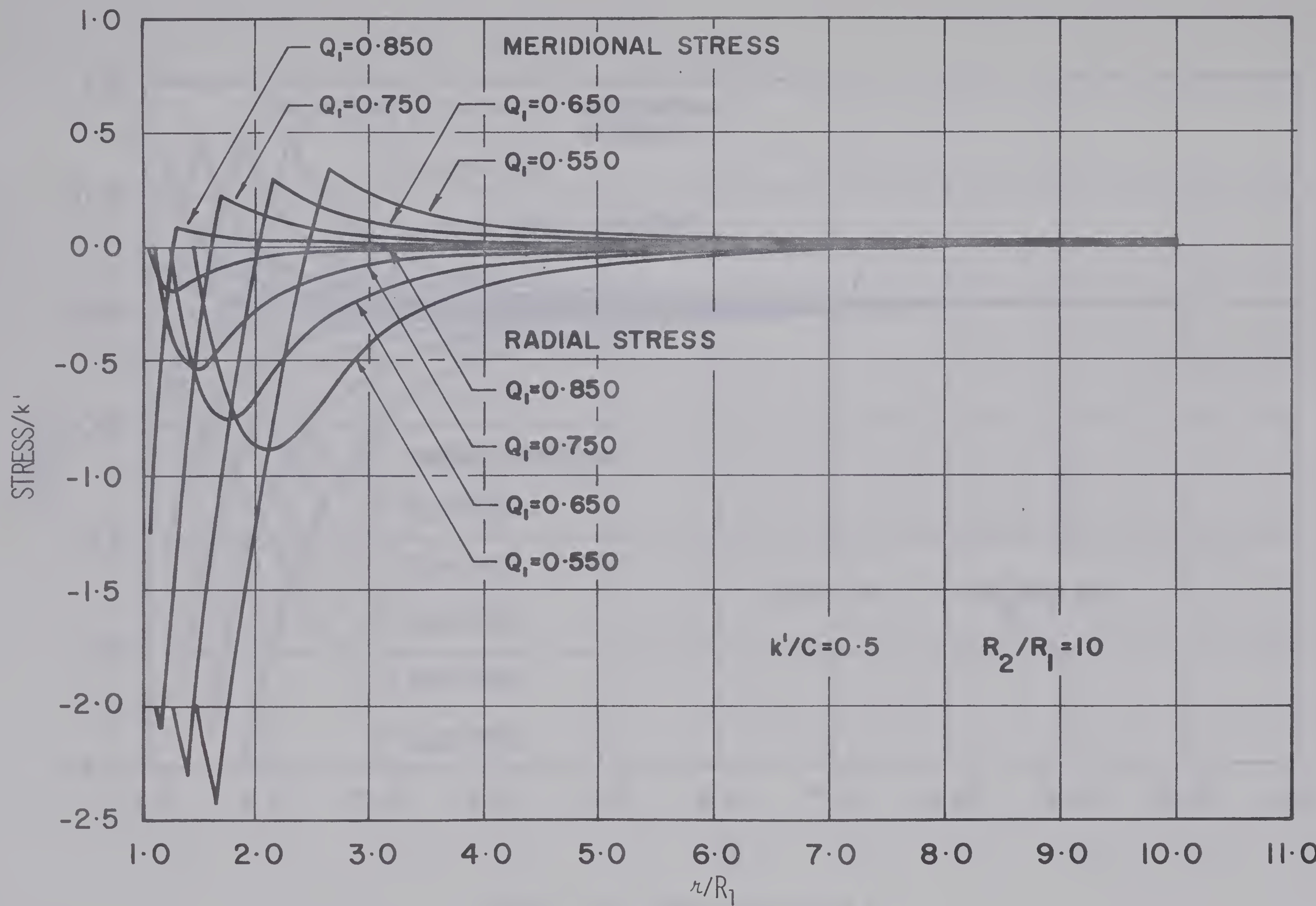



Figure 7.11 Residual Stresses

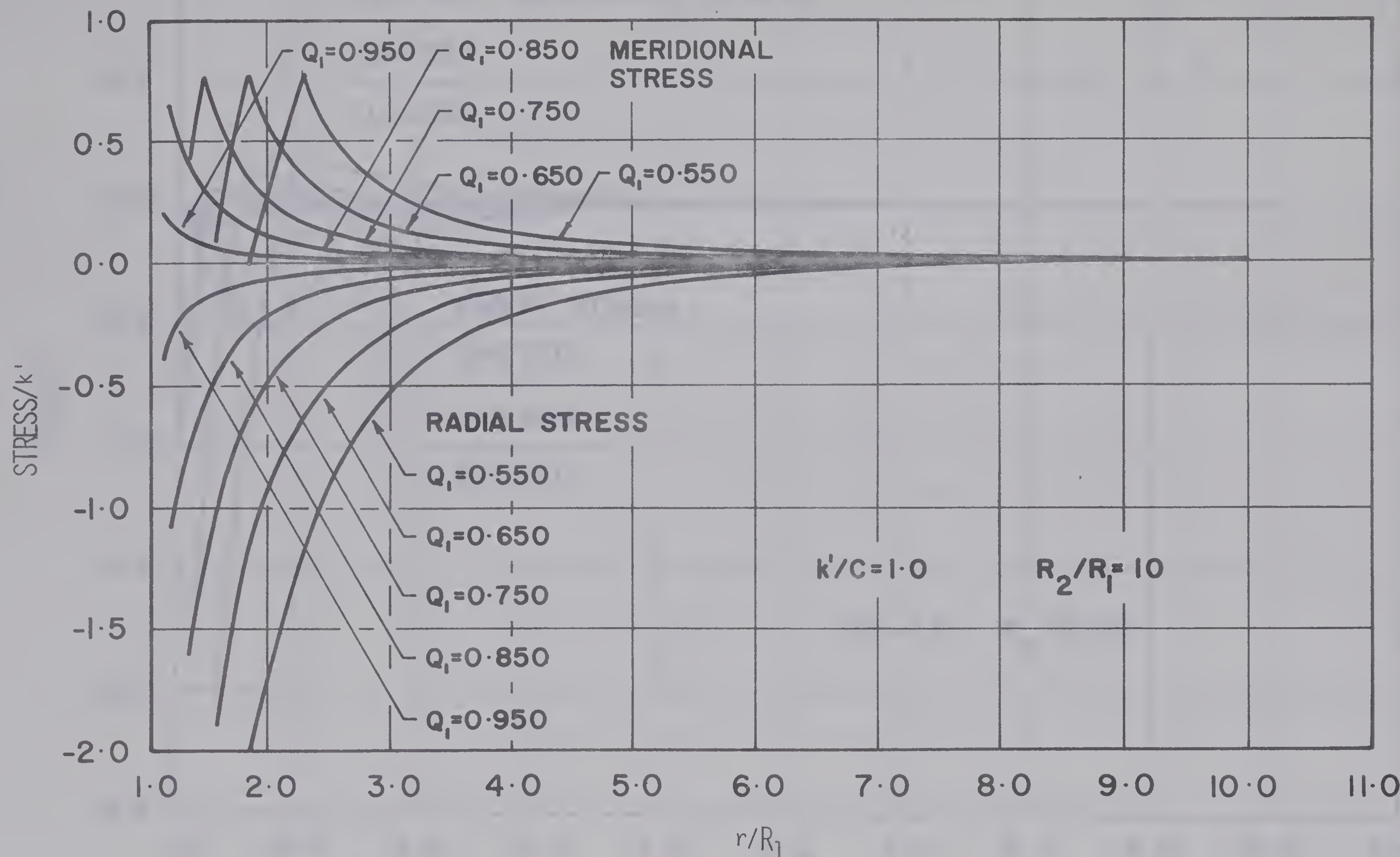
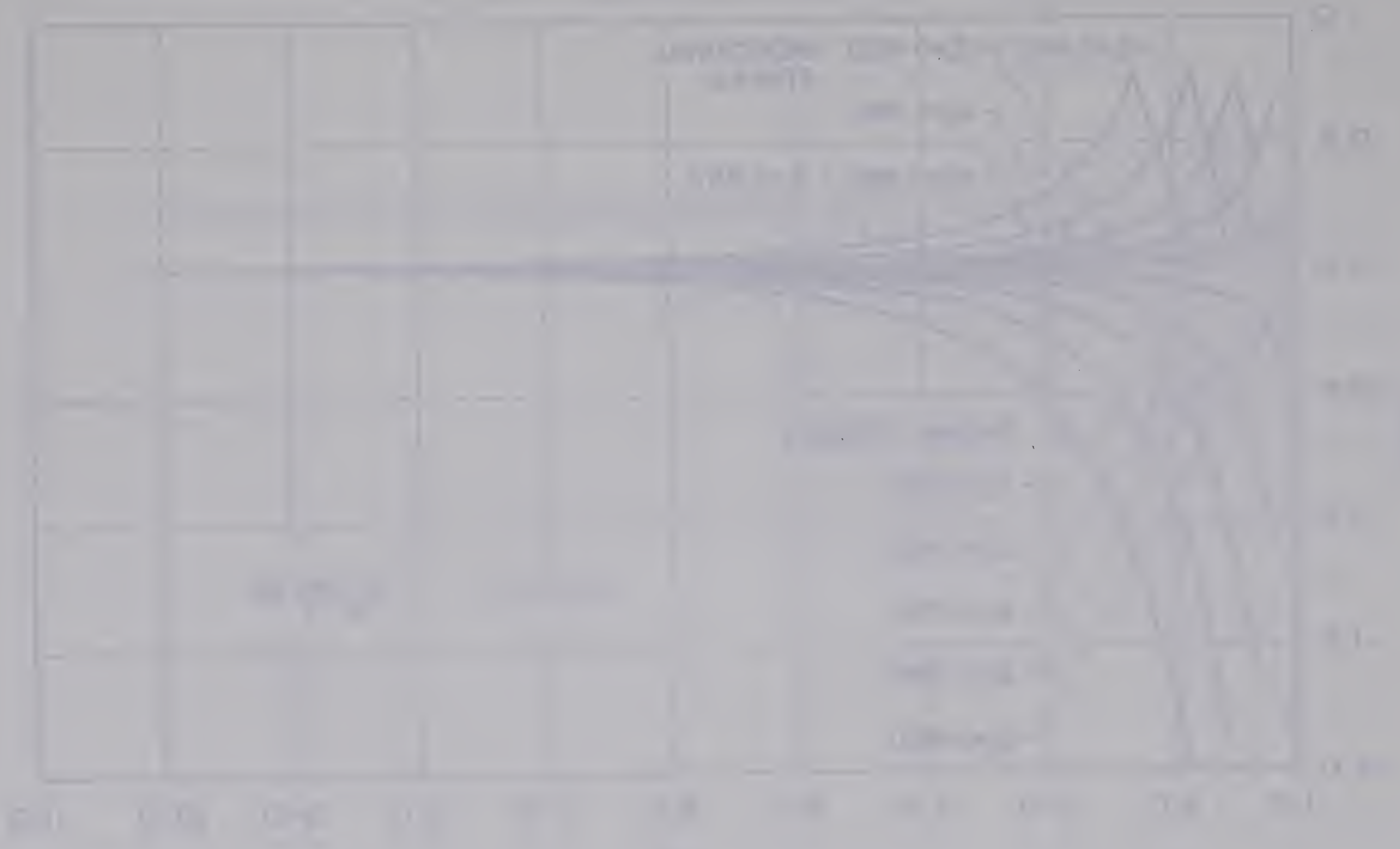



Figure 7.12 Stress Solution in C.3



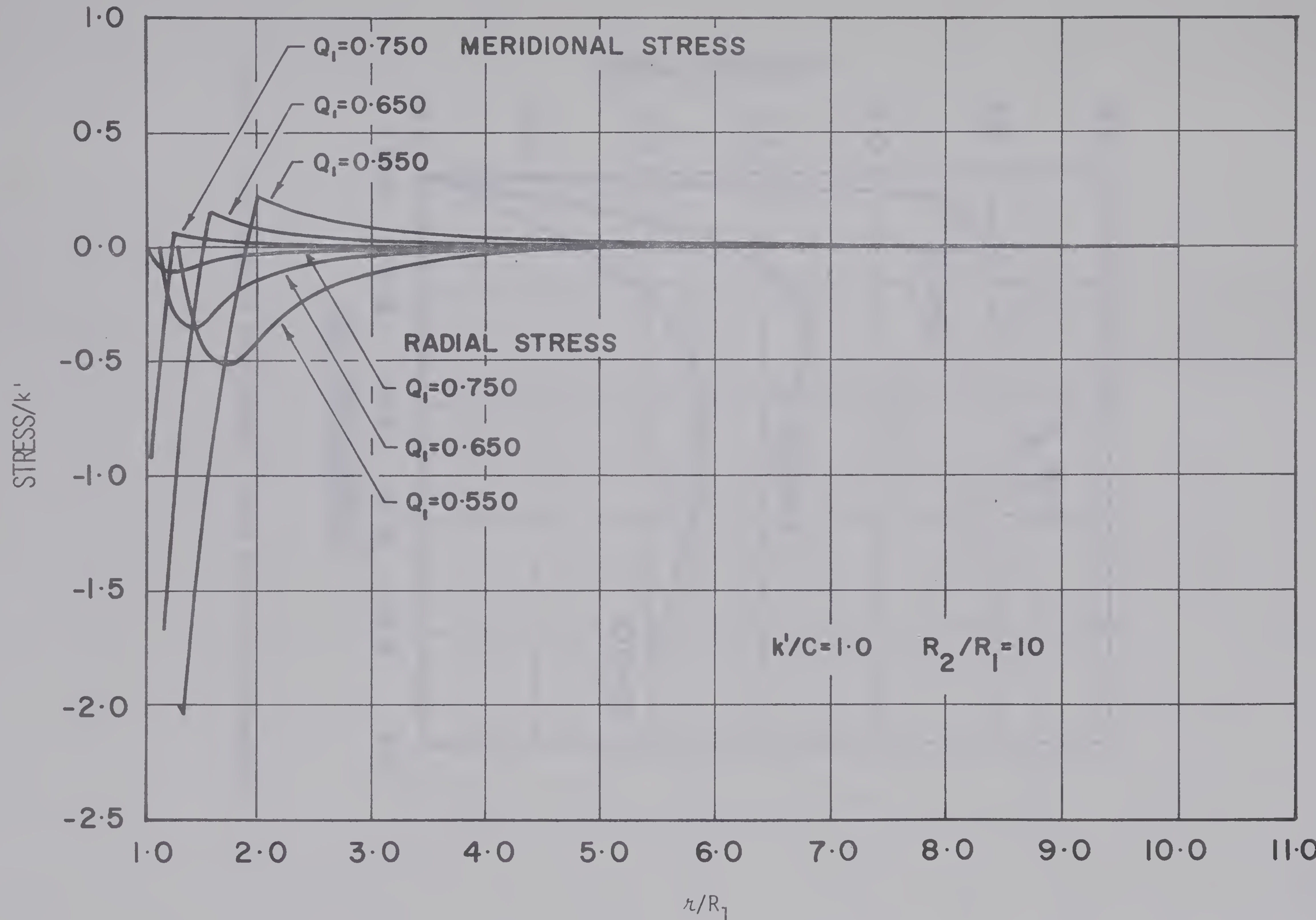
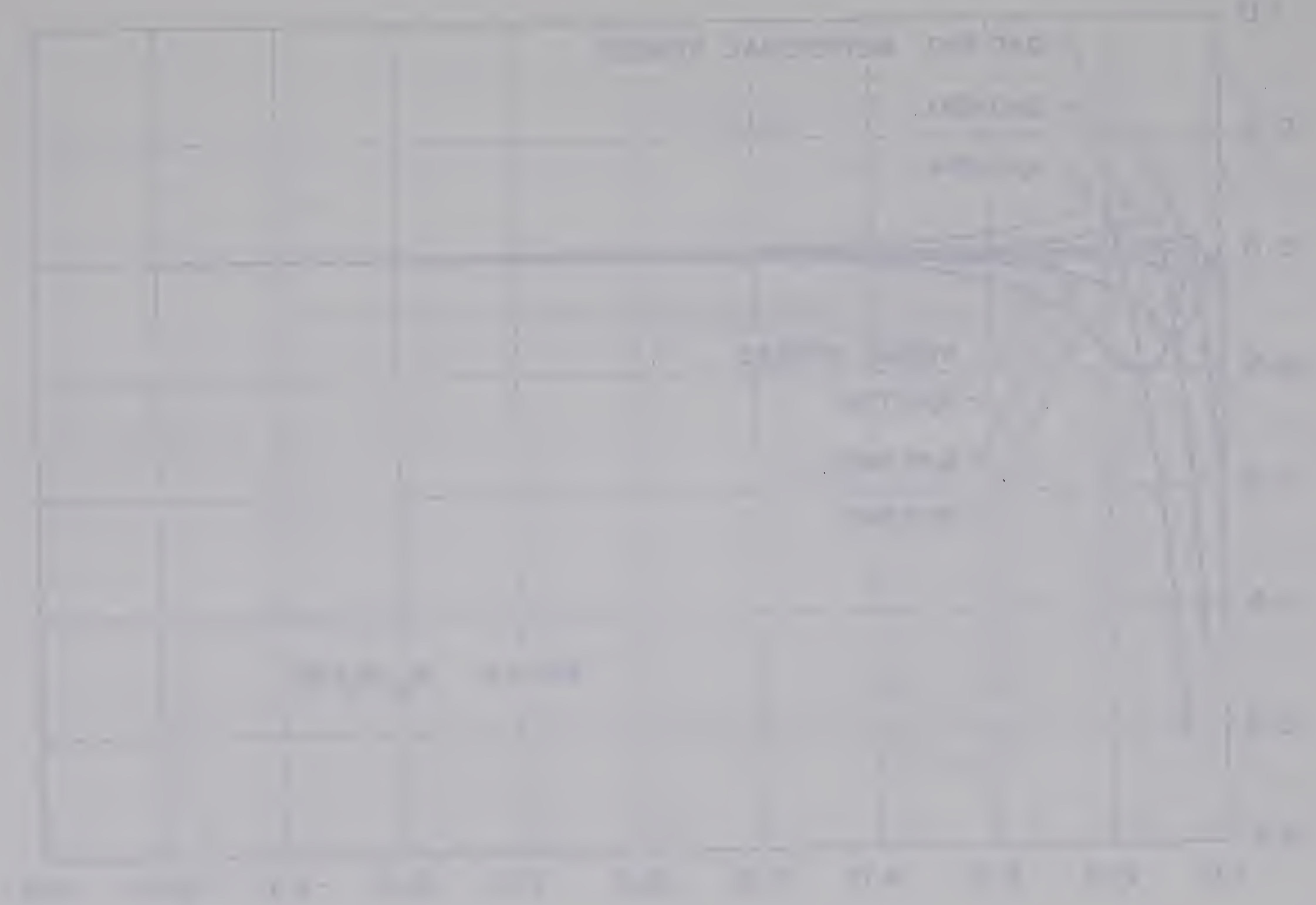



Figure 7.13 RESIDUAL STRESSES



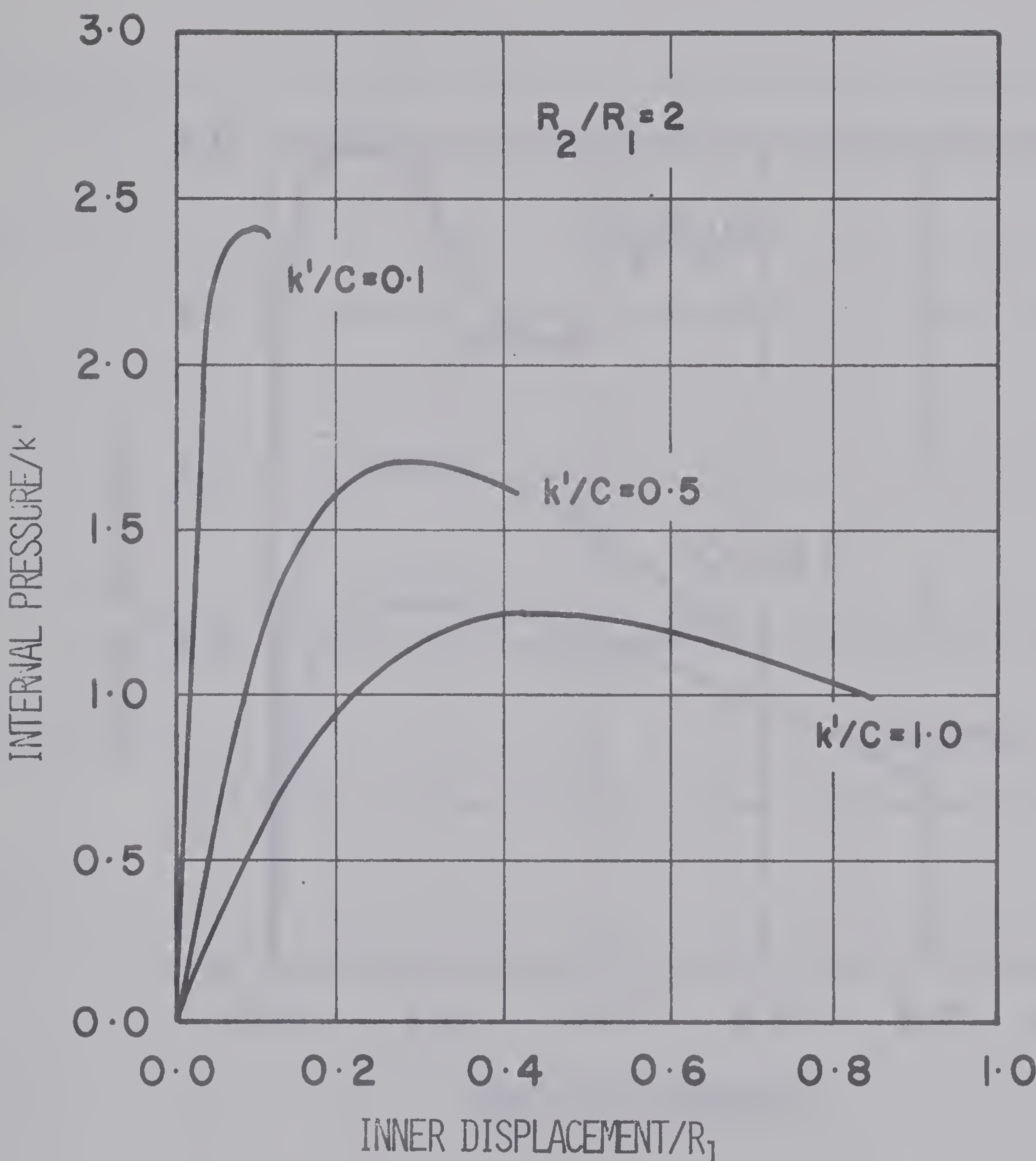



Figure 7.14 Internal Pressure/k' versus Inner Displacement/$R_1$

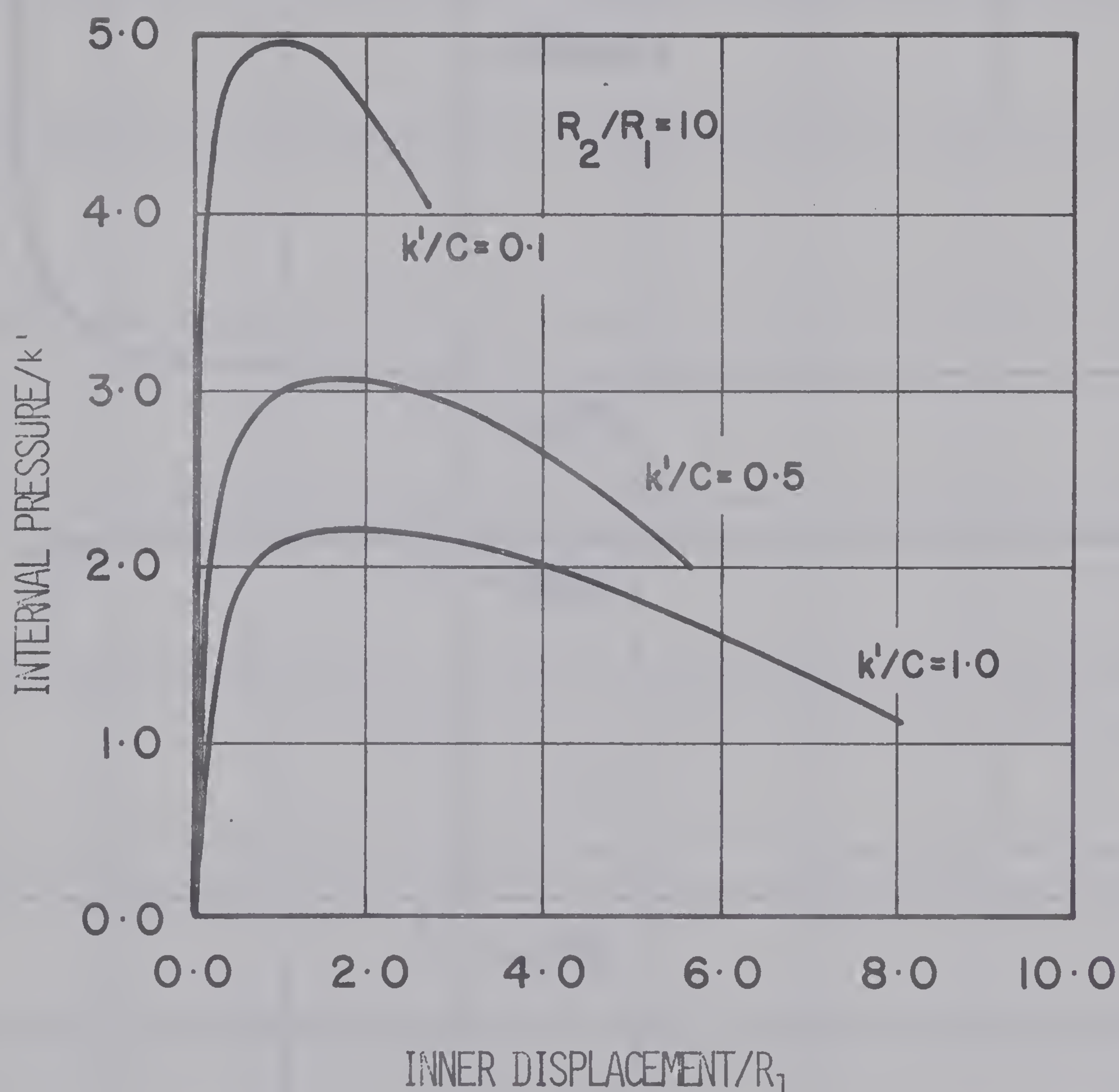



Figure 7.15 Internal Pressure/k' versus Inner Displacement/$R_1$

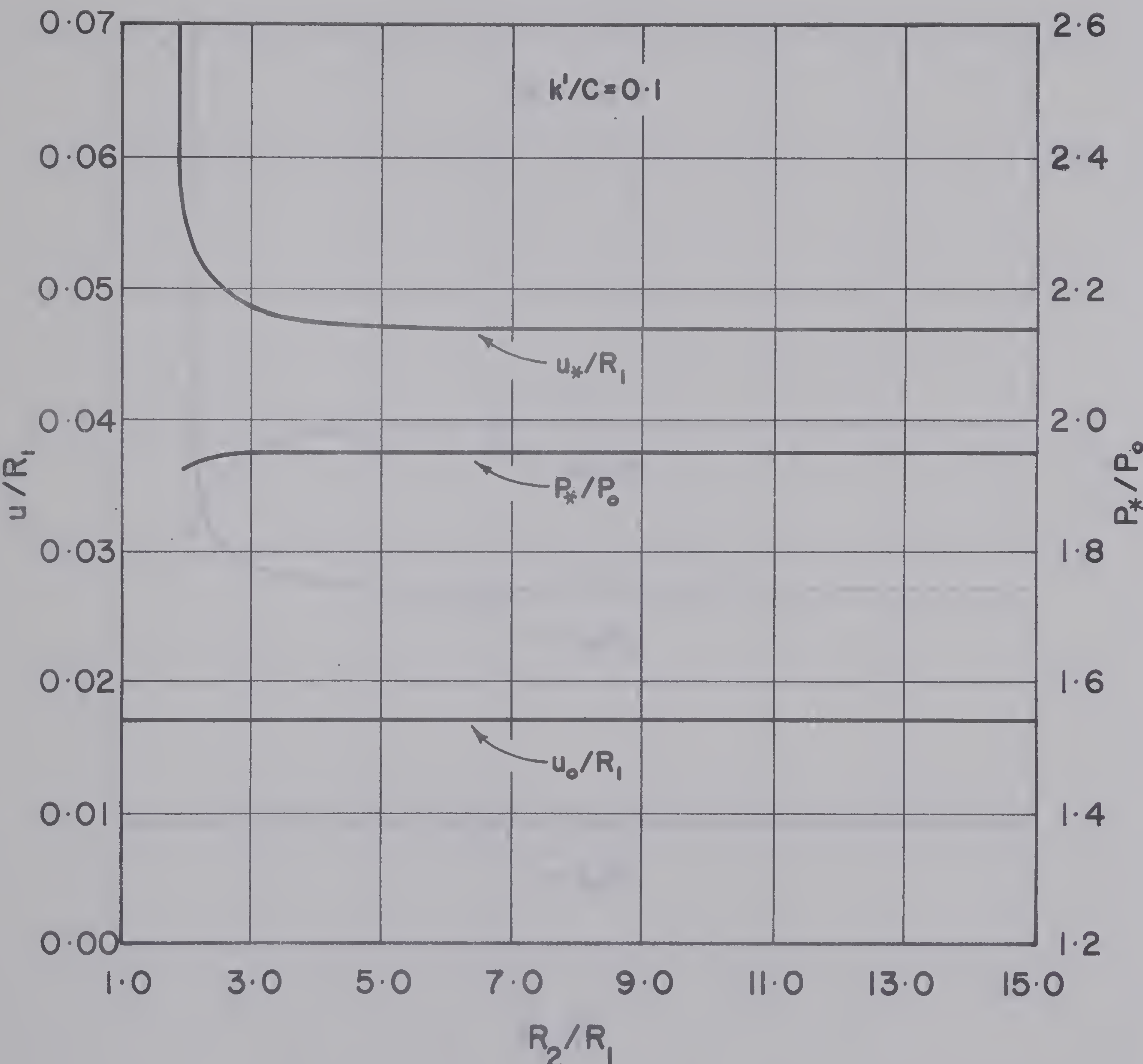



Figure 7.16 $u_o/R_1$, $u_*/R_1$ and $P_*/P_o$ versus $R_2/R_1$

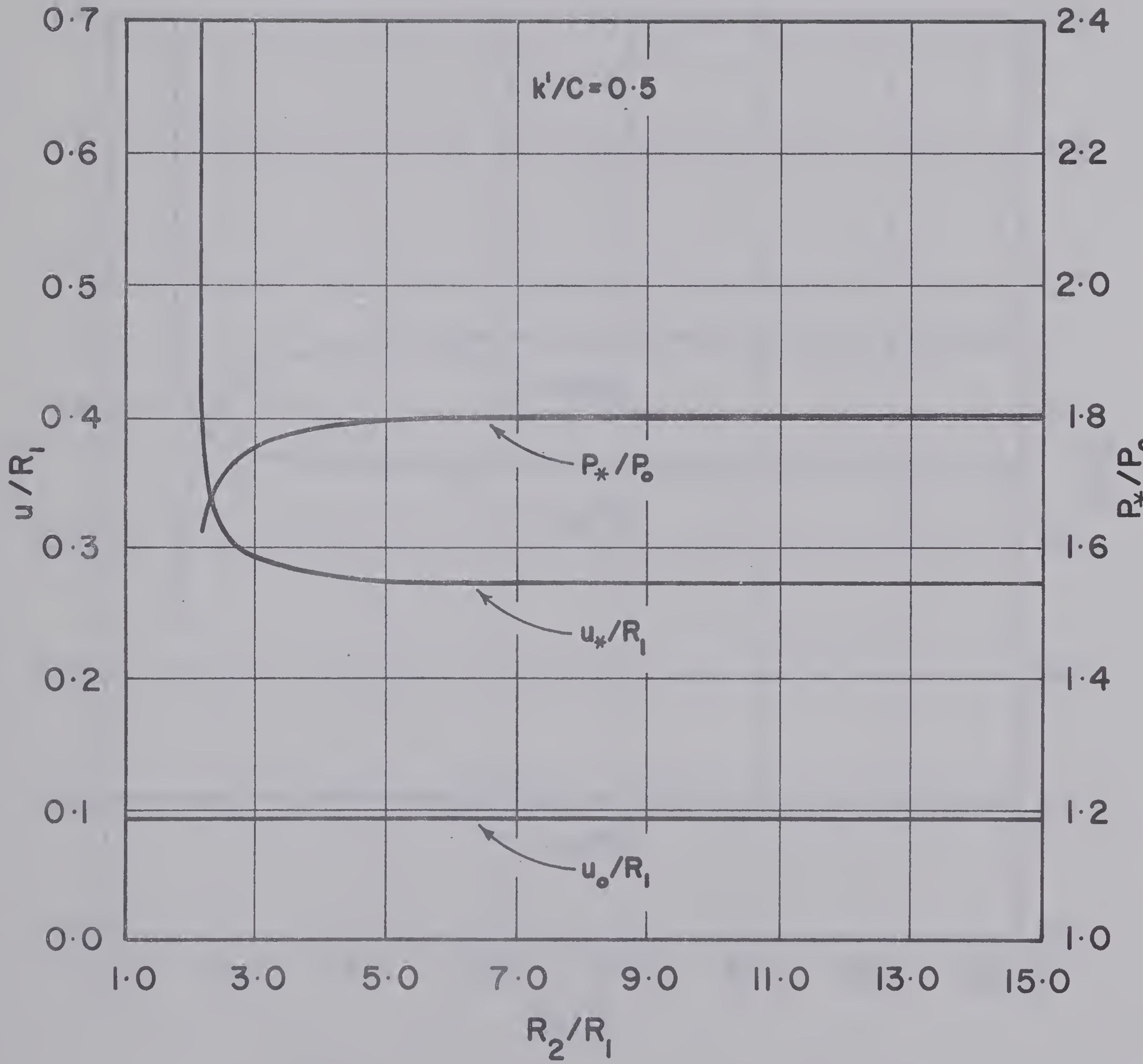
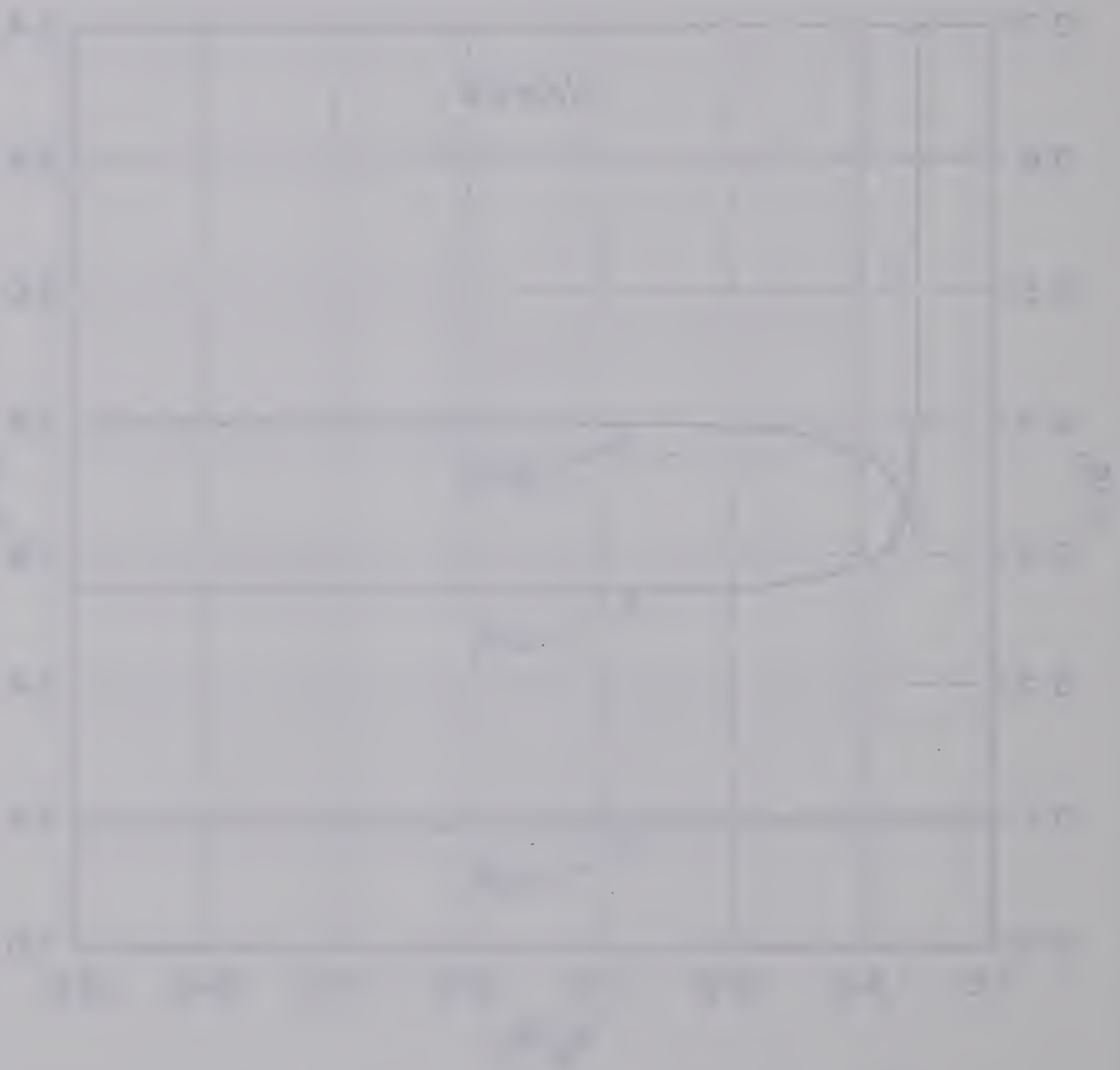



Figure 7.17 $u_0/R_1$, $u_*/R_1$, and $P_*/P_0$ versus $R_2/R_1$



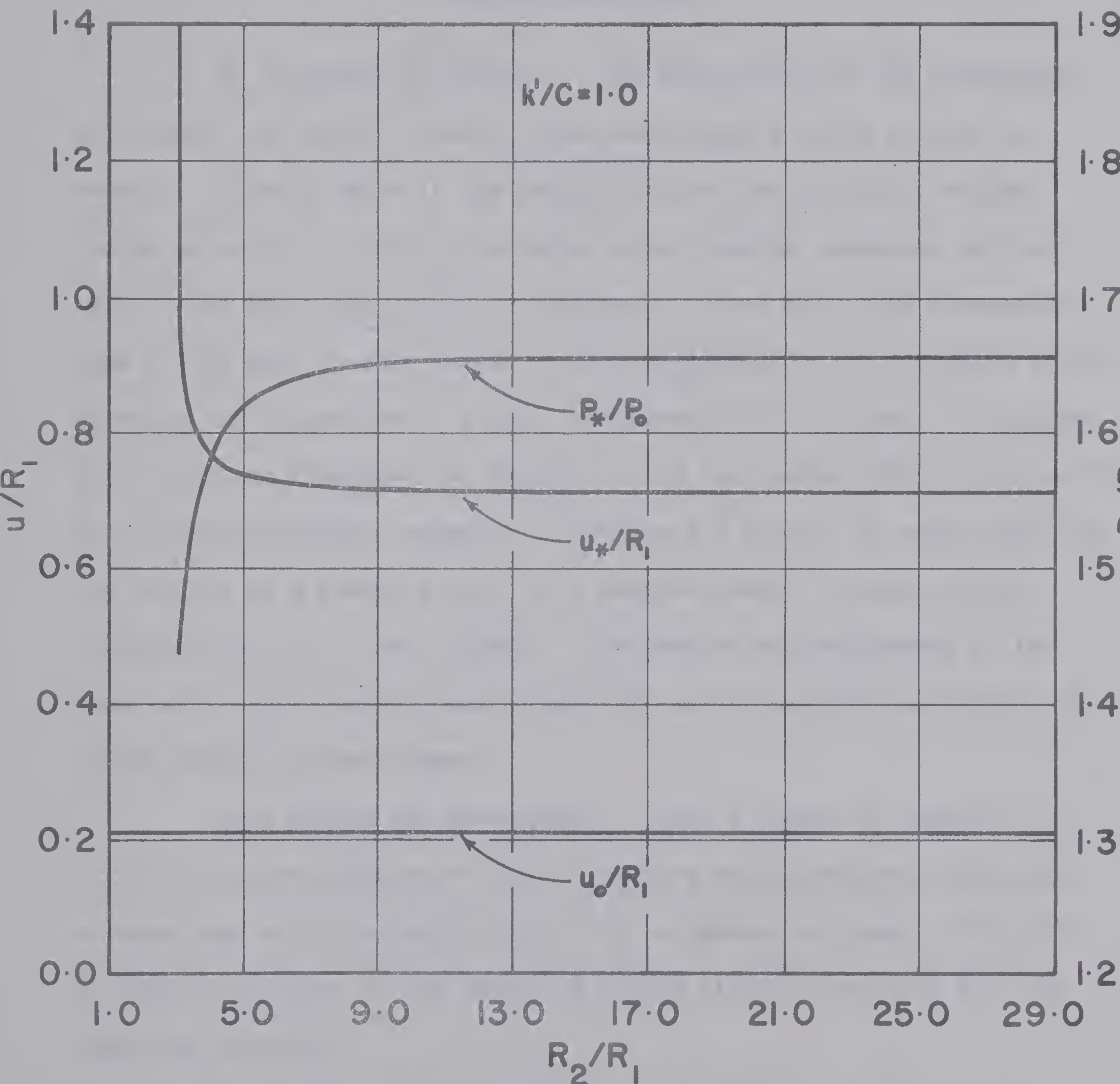
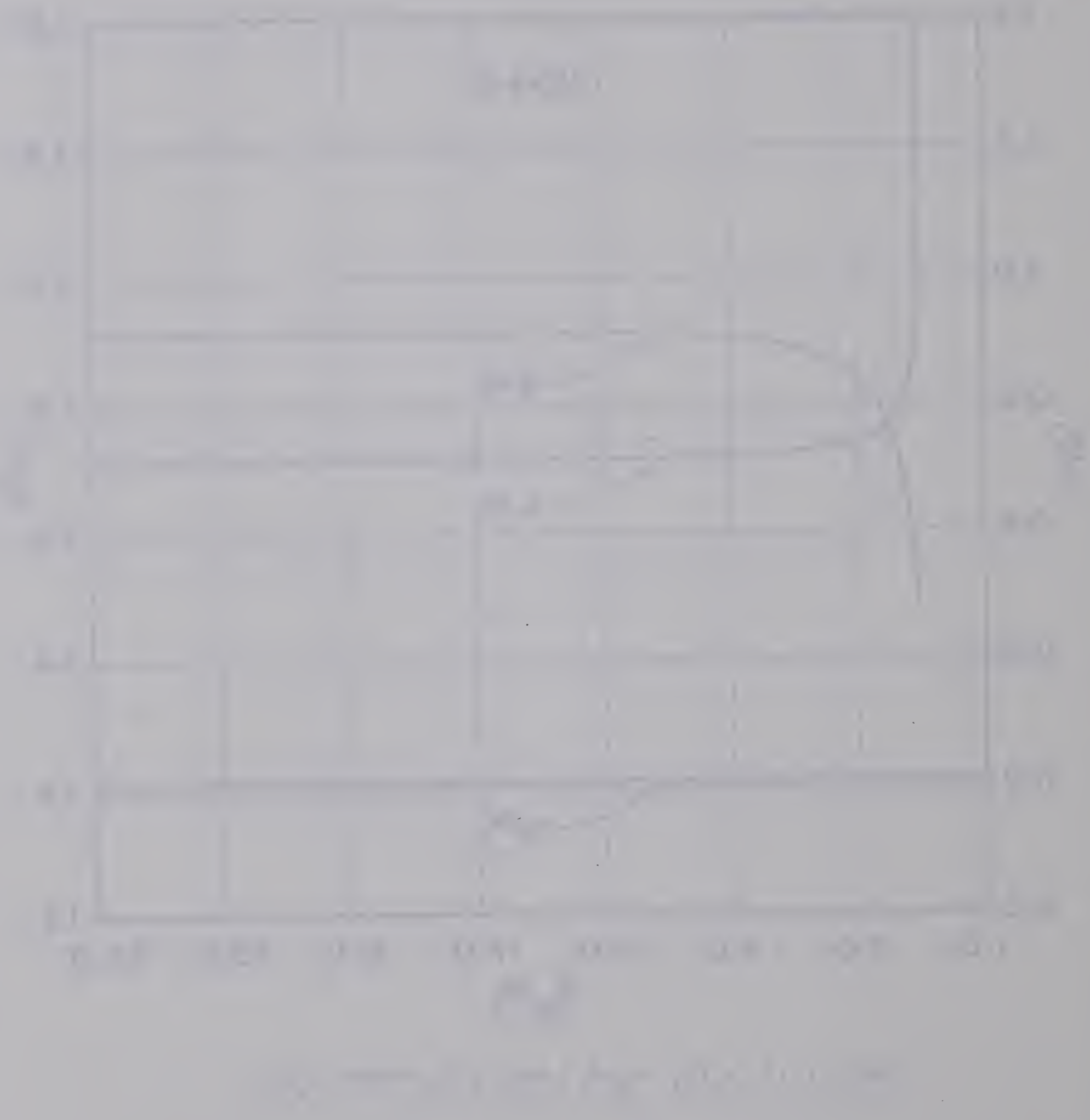



Figure 7.18 $u_o/R_1$, $u_*/R_1$, and $P_*/P_o$ versus $R_2/R_1$



# CHAPTER VIII

## CONCLUDING REMARKS

As indicated in Chapter I, the motivation for the development of a theory for elastic-plastic flow with finite elastic strains is twofold. Firstly there is the possibility of finite elastic volume change occurring in certain explosive metal forming processes and the work of Lee [25], [26], [27] is concerned in part with this phenomenon. Some of the work in this thesis is also applicable to this problem since although the assumption of elastic incompressibility is made in Chapters IV to VII, the kinematics in Chapter II and the thermo-elastic considerations for an elastic-plastic material in section 3.7 are not so restricted and may be used as a starting point of a general theory in which elastic incompressibility is not assumed. Furthermore the development of the flow rule (4.2.4) may be generalized [76] to include the possibility of finite elastic volume change.

Even though the development of such a theory is possible it is felt that the solution of actual boundary value problems using such a theory may be intractable since it is in general extremely difficult to obtain solutions in the theory of finite strain elasticity for compressible materials.

Two problems of interest for which such a theory may be required are the propagation of one dimensional elastic-plastic waves in

a plate due to an explosive charge on one of the plate surfaces and the explosive expansion of a spherical cavity in an infinite elastic-plastic medium. Lee [26], [27] has discussed the first problem and the latter has been considered by Hunter and Crozier [77] as well as in a comprehensive review by Hopkins [78]. Only small elastic strains are considered in these two works.

The development of a more general theory for elastic-plastic materials is also the result of indications that some elastomers and polymers, for certain ranges of temperature, exhibit permanent set after finite elastic shear strains with little if any elastic volume change. This is in contrast to the previously mentioned class of problems in which the reverse situation occurs, that is elastic-plastic flow with finite elastic volume change and small elastic shear strains. The possibility exists that an elastic-plastic theory in which finite elastic shear strains are considered may provide a useful approximation to the behavior of such materials and the work of Chapters IV to VII, in which elastic incompressibility is assumed, has been done with these materials in mind.

The assumption is made in Chapter IV that the materials considered are rate independent and satisfy Drucker's postulate. At present there is no evidence to indicate that for inelastic deformation the elastomers mentioned necessarily belong to this class of materials and although in the limited testing cited in Chapter I there was no indication that these materials were not in this class the question re-

mains the subject of further experimental work. Furthermore the two yield conditions discussed in Chapter IV are not based on experimental evidence and also remain to be investigated experimentally. Temperature is known to have a significant effect on the properties of these materials and any testing must certainly take this into account especially with regard to the assumption of rate independence. It is concluded therefore that considerable experimental work is required before the work of Chapters IV to VII may be linked with the physical behavior of any real materials.

Although the theory presented here is limited to perfect plasticity there is no difficulty in principle in including work-hardening in the development. There is no experimental evidence however to indicate that the work-hardening of the elastomers mentioned is or is not significant.

The neo-Hookean material has been chosen as the model for elastic behavior in Chapters IV to VII because it provides a reasonable approximation for the elastic behavior of some elastomers and because it simplifies the mathematical analysis. The inclusion in the theory of more elaborate models for incompressible elastic behavior such as the Mooney material, although likely to result in increased difficulty in solving actual problems, presents no difficulty in principle.

Elastic* solutions are known for a number of deformations

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*Here the term elastic refers to the general finite strain theory and not the classical Hookean theory.

which are included in the five families of deformations [73] which are possible in any isotropic incompressible elastic material** under the action of surface tractions only. Among these are the extension, inflation, and torsion (or shear) of a hollow cylinder, and the bending of a cuboid. An elastic-plastic solution to a problem cannot be found unless the elastic solution is known and these existing elastic solutions provide a number of problems which may be solvable using a finite strain elastic-plastic theory such as that developed here.

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**For a specific elastic material there may exist deformations which can be maintained by surface tractions alone but which are not included in these five families.

BIBLIOGRAPHY

1. Coulomb, C.A.: Essai sur une Application des Règles des Maxims et Minimis à Quelques Problèmes de Statique Relatifs à L'Architecture. Mèm. Prés. Par. Div. Savants, 7(1773), p. 343.

2. Tresca, H.: Mémoire sur L'Écoulement des Corps Solides Soumis à des Fortes Pressions. Comptes Rendus Acad. Sci. Paris, 59 (1864), p. 754.

3. Vénant, B. de Saint-: Mémoire sur L'Éstablissement des Équations Differentielles des Mouvement Interieurs Opérés dans les Corps Solides Ductiles au Déla des Limites où L'Élasticité Pourrait les Ramener à Leur Premier État. Comptes Rendus Acad. Sci. Paris, 70 (1870), p. 473.

4. Lévy, M.: Mémoire sur les Équations Génerales des Mouvements Interieurs des Corps Solides Ductiles au Déla des Limites où L'Élasticité Pourrait les Ramener à Leur Premier État. Comptes Rendus Acad. Sci. Paris, 70 (1870), p. 1323.

5. Mises, R. von: Mechanik der Festen Körper im Plastisch Deformablen Zustand. Göttinger Nachr., Math. Phys. Klasse (1913), p. 582.

6. Mises, R. von: Mechanik der Plastischen Formänderung von Kristallen. Zeit. Ang. Math. Mech., 8(1928), p. 161.

7. Hencky, H.: Zur Theorie Plastisher Deformationen und der Hierdurch im Material Hervorgeruffenen Nachspannungen. Zeit. Ang. Math. Mech., 4 (1924), 323.

8. Handelman, G.H., C.C. Lin, and W. Prager: On the Mechanical Behavior of Metals in the Strain-Hardening Range. Quart. Appl. Math., 4 (1947), p. 397.

9. Hill, R.: The Mathematical Theory of Plasticity, Oxford University Press, 1950, p. 47.

10. Prandtl, R.: Spannungsverteilung in Plastischen Körpern. Proc. 1st Int. Congr. Appl. Mech., Delft, 1924, p. 43.

11. Reuss, A.: Berücksichtigung der Elastichen Formänderungen in der Plastizitätstheorie. Zeit. Ang. Math. Mech. 10 (1930), p. 266.

12. Hill, R.: The Mathematical Theory of Plasticity. Oxford University Press, 1950.

13. Prager, W. and P.G. Hodge: The Theory of Perfectly Plastic Solids. John Wiley and Sons, 1951.

14. Sokolovsky, V.V.: Theory of Plasticity, (in Russian). Moscow, 1945.

15. Thomas, T.Y.: Plastic Flow and Fracture in Solids. Academic Press, New York, 1961.

16. Prager, W.: An Introduction to Plasticity. Addison-Wesley, 1959.

17. Goddier, J.H. and P.G. Hodge: Elasticity and Plasticity. John Wiley and Sons, New York, 1958.

18. Koiter, W.T.: General Theorems for Elastic-Plastic Solids. Progress in Solid Mechanics, Amsterdam, North-Holland, 1960, p. 167.

19. Freudenthal, A.M. and H. Geirenger: The Mathematical Theories of the Inelastic Continuum. Handbuck der Physik 6(1958), Springer, Berlin.

20. Thomas, T.Y.: Combined Elastic and Prandtl-Reuss Stress Strain Relations. Proc. Nat. Acad. Sci. U.S., 41 (1955), p. 720.

21. Sedov, L.I.: An Introduction to the Mechanics of a Continuous Media. Addison-Wesley, 1965, pp. 230-259.

22. Pipkin, A.C. and R.S. Rivlin: Mechanics of Rate Independent Materials. Zeit. Ang. Math. Phys. 16 (1965), p. 313.

23. Green, A.E. and P.M. Naghdi: A General Theory of an Elastic-Plastic Continuum. Arch. Rat. Mech. Anal. 18 (1965), p. 251.

24. Green, A.E. and P.M. Naghdi: A Thermodynamic Development of Elastic-Plastic Continua. I.U.T.A.M. Symp.,Vienna, 1966, ed. Parkus, H. and R.I. Sedov.

25. Lee, E.H.: Elastic-Plastic Waves of One Dimensional Strain. Proc. 5th U.S. National Congress of Applied Mechanics, 1966, p. 405.

26. Lee, E.H., and D.T. Liu: Finite Strain Elastic-Plastic Theory, I.U.T.A.M. Symp., Vienna, 1966, ed. Parkus, H. and R.I. Sedov.

27. Lee, E.H.: Elastic-Plastic Deformations at Finite Strains. J. Appl. Mech., March, 1969.

28. Freund, L.B.: Constitutive Equations for Elastic-Plastic Materials at Finite Strain. Int. Journal of Solids and Structures, 6 (1970), p. 1193.

29. Sokolnikoff, I.S.: Mathematical Theory of Elasticity. McGraw-Hill, 1956, p. 67.

30. Drucker, D.C.: A More Fundamental Approach to Plastic Stress-Strain Relations. Proc. 1st U.S. Natl. Congr. Appl. Mech. (1951), p. 487.

31. Bridgman, P.W.: Flow and Fracture. Trans. Am. Inst. Min.Met. Eng., 162 (1945), p. 569.

32. Thomas, T.Y.: Interdependence of the yield Condition and the Stress-Strain Relations for Plastic Flow. Proc. Nat. Acad. Sci. U.S., 40 (1955), p. 593.

33. Leigh, D.C.: Non-Linear Continuum Mechanics. McGraw-Hill, 1968, p. 143.

34. Truesdell, C. and W. Noll: Handbuch der Physik. III/3 . The Non-Linear Field Theories of Mechanics, Springer-Verlag, 1965, p. 44.

35. Noll, W.: On the Continuity of Solid and Fluid States. J. Rat. Mech. Anal., 4 (1955), pp. 3-81.

36. Oldroyd, J.G.: On the Formulation of Rheological Equations of State. Proc. Roy. Soc. London, No. 200 A, (1950), p. 523.

37. Cotter, B. and R.S. Rivlin: Tensors Associated with Time-Dependent Stress. Quart. Appl. Math., 13 (1955), p. 177.

38. Jaumann, G: Die Grundlagen der Bewegungslehre von Eienem Modernen Standpunkte Aus. Leipzig (1905). Also Sitzgeber. Akad. Wiss. Wien. (IIa), 120 (1911), p. 385.

39. Prager, W.: An Elementary Discussion of Definitions of Stress Rate. Quart. Appl. Math., 18 (1961), p. 403.

40. Thomas, T.Y.: Plastic Flow and Fracture in Solids. Academic Press, New York, 1961, p. 83-87.

41. Danyluk, H.T.: The Flow of an Incompressible Elastic-Perfectly Plastic Solid. Ph.D. Thesis, University of Alberta, Edmonton, 1969.

42. Haddow, J.B. and H.T. Danyluk: The Flow of an Incompressible Elastic-Perfectly Plastic Solid. Acta. Mech. 5 (1968), p. 14.

43. Danyluk, H.T. and J.B. Haddow: Elastic-Plastic Flow Through a Converging Conical Channel. Acta. Mech., 7 (1968), p. 35.

44. Backman, M.E.: Form for the Relation Between Stress and Finite Elastic and Plastic Strains Under Impulsive Loading. J. Appl. Phys., 35 (1964), p. 2524.

45. Eringen, A.C.: Nonlinear Theory of Continuous Media. McGraw-Hill, 1962, p. 44.

46. Eringen, A.C.: Mechanics of Continua. John Wiley and Sons, 1967, p. 8.

47. Truesdell, C. and W. Noll: Handbuch der Physik. III/3 . The Non-Linear Field Theories of Mechanics, Springer-Verlag, 1965, p. 294.

48. Aris, R.: Vectors, Tensors and the Basic Equations of Fluid Mechanics. Prentice-Hall, 1962, p. 183.

49. Eringen, A.C.: Mechanics of Continua. John Wiley & Sons, 1967, p. 11 and p. 40.

50. Prager, W.: Introduction to Mechanics of Continua. Ginn and Co., 1961, p. 198.

51. Green, A.E., and W. Zerna: Theoretical Elasticity, Oxford University Press, Second Edition, 1968, pp. 68-70.

52. Aris, R.: Vectors, Tensors and the Basic Equations of Fluid Mechanics. Prentice-Hall, 1962, p. 171.

53. Green, A.E. and W. Zerna: Theoretical Elasticity, Oxford University Press, Second Edition, 1968, pp. 63-64.

54. Mooney, M.: A Theory of Large Elastic Deformation. J. Appl. Phys., 11 (1940), p. 582.

55. Treloar, L.R.G.: The Physics of Rubber Elasticity. Oxford University Press, 1958, Chapter IV.

56. Leigh, D.C.: Non-linear Continuum Mechanics. McGraw-Hill, 1968, pp. 136-138.

57. Kestin, J.: A Course in Thermodynamics. Blaisdell Pub. Co., 1966, p. 536.

58. Green, A.E. and W. Zerna: Theoretical Elasticity. Oxford University Press, Second Edition, 1968, pp. 72-76.

59. Prager, W.: Introduction to Mechanics of Continua. Ginn and Co. , 1961, p. 200.

60. Kestin, J.: On the Application of the Principles of Thermodynamics to Strained Solid Materials. I.U.T.A.M. Symposium, ed. H. Parkus and L.I. Sedov, Vienna, 1966.

61. Pearson, C.E.: Theoretical Elasticity. Harvard University Press, 1959, p. 164.

62. Timoshenko, S. and J.M. Gere: Theory of Elastic Stability. McGraw-Hill, Second Edition, 1961, p. 76.

63. Rivlin, R.S.: Large Elastic Deformations of Isotropic Elastic Materials IV, Further Developments of the General Theory. Phil. Trans. A, 241, p. 379.

64. Green, A.E. and W. Zerna: Theoretical Elasticity. Oxford University Press, Second Edition, 1968, pp. 79-80.

65. Babuska, I., M. Prager, and E. Vitasek: Numerical Processes in Differential Equations. SNTL, Publishers of Technical Literature, Prague, 1966.

66. Collatz, L.: The Numerical Treatment of Differential Equations. Springer-Verlag, New York, 1966, pp. 61-71.

67. Lapidus, L.: Digital Computation for Chemical Engineers. McGraw-Hill, 1962, pp. 56-62.

68. Lamé : Leçons sur la Théorie ... de L'Élasticité. Paris, 1852.

69. Green, A.E. and R.T. Shield: Finite Elastic Deformation of Incompressible Isotropic Bodies. Proc. Roy. Soc. London, A202, 1950, p. 407.

70. Hill, R.: The Mathematical Theory of Plasticity. Oxford University Press, 1950, pp. 97-104.

71. Hill, R.: General Features of Plastic-Elastic Problems as Exemplified by Some Particular Solutions. J. Appl. Mech. 16 (1949), p. 295.

72. Isaacson, E. and H.B. Keller: Analysis of Numerical Methods. John Wiley & Sons, 1966, p. 102.

73. Truesdell, C.: The Elements of Continuum Mechanics. Springer-Verlag, 1966, pp. 117-128.

74. Lapidus, L: Digital Computation for Chemical Engineers. McGraw-Hill, 1962, pp. 36-41.

75. Handbook of Mathematical Functions, U.S. Dept. of Commerce, National Bureau of Standards, Applied Mathematics Series, 1968, p. 916.

76. Haddow, J.B. and Hrudey, T.M.: The Yield Condition and Flow Rule for a Metal Subjected to Finite Elastic Volume Change. Submitted to the J. of Basic Eng.

77. Hunter, S.C. and R.J.M. Crozier: Similarity Solution for the Rapid Uniform Expansion of a Spherical Cavity in a Compressible Elastic-Plastic Solid. Quart. J. Mech. Appl. Math., 21, 4, 1968, p. 467.

78. Hopkins, H.G.: Dynamic Expansion of Spherical Cavities in Metals. Progress in Solid Mechanics, North-Holland Pub., Vol. I, 1960, p.85.

# APPENDIX A

## A THEOREM USED IN SECTION 4.4

The following theorem which is used to simplify the plastic flow rule developed in Chapter IV is now proved.

Theorem: If $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ are square symmetric matrices with $\underset{\sim}{B}$ positive definite, and the principal directions of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ coincide, and if

$$\underset{\sim}{A} = \underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{B}^{-1} + \underset{\sim}{B}^{-1}\underset{\sim}{C}\underset{\sim}{B} \qquad (A.1)$$

then the principal directions of the matrix $\underset{\sim}{C}$ coincide with those of $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

Proof:

Let $\underset{\sim}{x}$ ( a column vector) be in the direction of any one of the eigenvectors of $\underset{\sim}{A}$, and therefore by hypothesis it is in the direction of one of the eigenvectors of $\underset{\sim}{B}$. Thus there exist scalars $\lambda$ and $\Lambda$ such that

$$\underset{\sim}{A}\underset{\sim}{x} = \Lambda\underset{\sim}{x} \qquad (A.2)$$

$$\underset{\sim}{B}\underset{\sim}{x} = \lambda\underset{\sim}{x} \qquad (A.3)$$

and

$$\underset{\sim}{B}^{-1}\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x} \ . \qquad (A.4)$$

From equations (A.1), (A.2), (A.3), and (A.4)

$$\underset{\sim}{A}\underset{\sim}{x} = \Lambda\underset{\sim}{x}$$

$$= (\underset{\sim}{B}\underset{\sim}{C}\underset{\sim}{B}^{-1} + \underset{\sim}{B}^{-1}\underset{\sim}{C}\underset{\sim}{B})\underset{\sim}{x}$$

$$= (\frac{1}{\lambda}\underset{\sim}{B}\underset{\sim}{C} + \lambda\underset{\sim}{B}^{-1}\underset{\sim}{C})\underset{\sim}{x}$$

$$= (\frac{1}{\lambda}\underset{\sim}{B} + \lambda\underset{\sim}{B}^{-1})\underset{\sim}{C}\underset{\sim}{x} \; . \tag{A.5}$$

There exists a scalar $\alpha$ and a vector $\underset{\sim}{y}$ such that

$$\underset{\sim}{C}\underset{\sim}{x} = \alpha\underset{\sim}{x} + \underset{\sim}{y}$$

where

$$\underset{\sim}{y}^T\underset{\sim}{x} = 0 \; . \tag{A.6}$$

Thus equation (A.5) becomes

$$\Lambda\underset{\sim}{x} = (\frac{1}{\lambda}\underset{\sim}{B} + \lambda\underset{\sim}{B}^{-1})(\alpha\underset{\sim}{x} + \underset{\sim}{y})$$

$$= (\frac{\alpha}{\lambda}\underset{\sim}{B}\underset{\sim}{x} + \lambda\alpha\,\underset{\sim}{B}^{-1}\underset{\sim}{x}) + (\frac{1}{\lambda}\underset{\sim}{B}\underset{\sim}{y} + \lambda\underset{\sim}{B}^{-1}\underset{\sim}{y})$$

$$= 2\alpha\underset{\sim}{x} + (\frac{1}{\lambda}\underset{\sim}{B}\underset{\sim}{y} + \lambda\underset{\sim}{B}^{-1}\underset{\sim}{y}) \; . \tag{A.7}$$

Premultiplying equation (A.7) by $\lambda \underset{\sim}{y}^T$ and using the condition (A.6) gives

$$\underset{\sim}{y}^T \underset{\sim}{B} \underset{\sim}{y} + \lambda^2 \underset{\sim}{y}^T \underset{\sim}{B}^{-1} \underset{\sim}{y} = 0 ,$$

and since $\underset{\sim}{B}$ is positive definite $\underset{\sim}{y}$ must be the null vector so that

$$\underset{\sim}{C} \underset{\sim}{x} = \alpha \underset{\sim}{x} .$$

That is, $\underset{\sim}{x}$ is an eigenvector of $\underset{\sim}{C}$. Consequently each of the eigenvectors of $\underset{\sim}{A}$ or $\underset{\sim}{B}$ has the same direction as one of the eigenvectors of C so that $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ are coaxial.

# APPENDIX B

## EQUILIBRIUM EQUATIONS IN CYLINDRICAL POLAR COORDINATES

Consider the cylindrical polar coordinate system shown in Figure B.1.

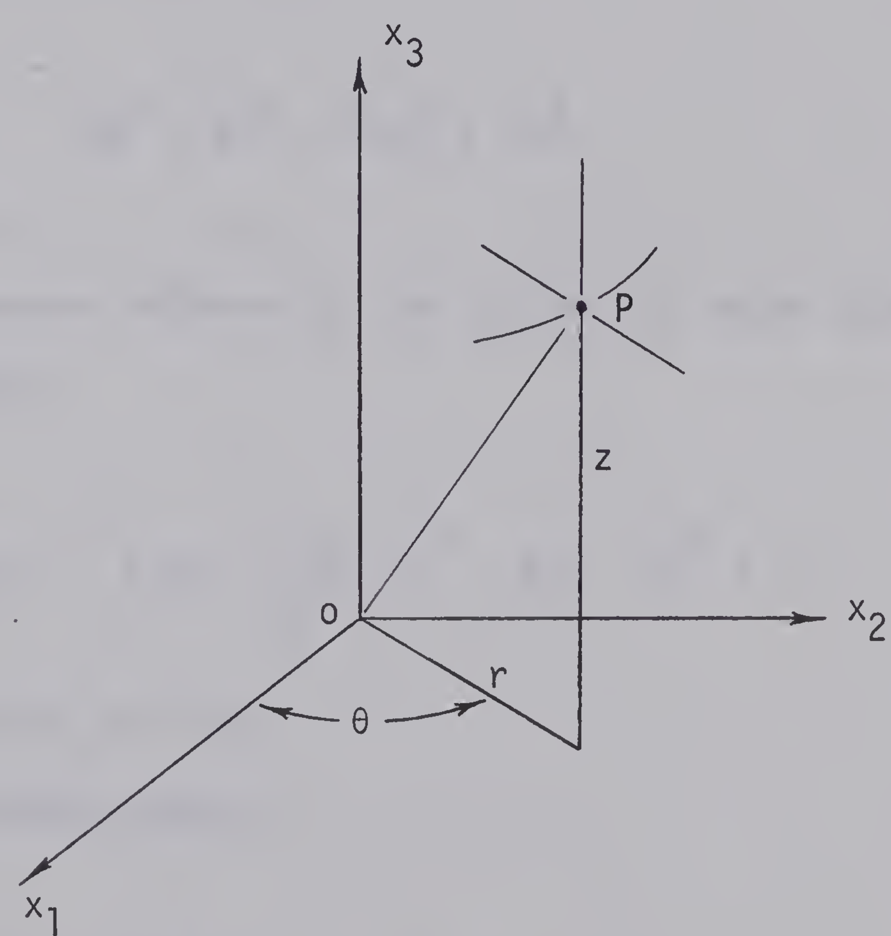


Figure B.1

Cylindrical Polar Coordinate System

Let the quantities $\xi_i$ be defined by

$$r = \xi_1 ,$$

$$\theta = \xi_2 ,$$

and $$z = \xi_3 .$$

An element of length ds is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 ,$$

so that the metric tensor referred to the cylindrical polar coordinates $\xi_i$ has components

$$g^{11} = g_{11} = 1 , \; g_{22} = \frac{1}{g^{22}} = r^2 , \; g_{33} = g^{33} = 1 ,$$

and all other components are zero.

The Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} (g_{km,j} + g_{jk,m} - g_{mj,k})$$

are thus

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} , \; \Gamma^1_{22} = - r , \tag{B.1}$$

and all other $\Gamma^i_{jk}$ are equal to zero.

Cauchy's first equation of motion (3.3.3) reduces to the equilibrium equation

$$\tau^{ij}{}_{;i} = 0$$

for a quasi-static process in the absence of body forces. That is

$$\tau^{ij}{}_{,i} + \Gamma^{i}_{ki}\tau^{kj} + \Gamma^{j}_{ki}\,\tau^{ik} = 0$$

so that using equations (B.1) the equilibrium equations in cylindrical polar coordinates are

$$\tau^{11}{}_{,1} + \tau^{21}{}_{,2} + \tau^{31}{}_{,3} + \frac{1}{r}\tau^{11} - r\tau^{22} = 0 \ ,$$

$$\tau^{12}{}_{,1} + \tau^{22}{}_{,2} + \tau^{32}{}_{,3} + \frac{3}{r}\tau^{12} = 0 \ ,$$

$$\tau^{13}{}_{,1} + \tau^{23}{}_{,2} + \tau^{33}{}_{,3} + \frac{1}{r}\tau^{13} = 0 \ .$$

APPENDIX C

## EQUILIBRIUM EQUATIONS IN SPHERICAL POLAR COORDINATES

Consider the spherical polar coordinate system in Figure C.1.

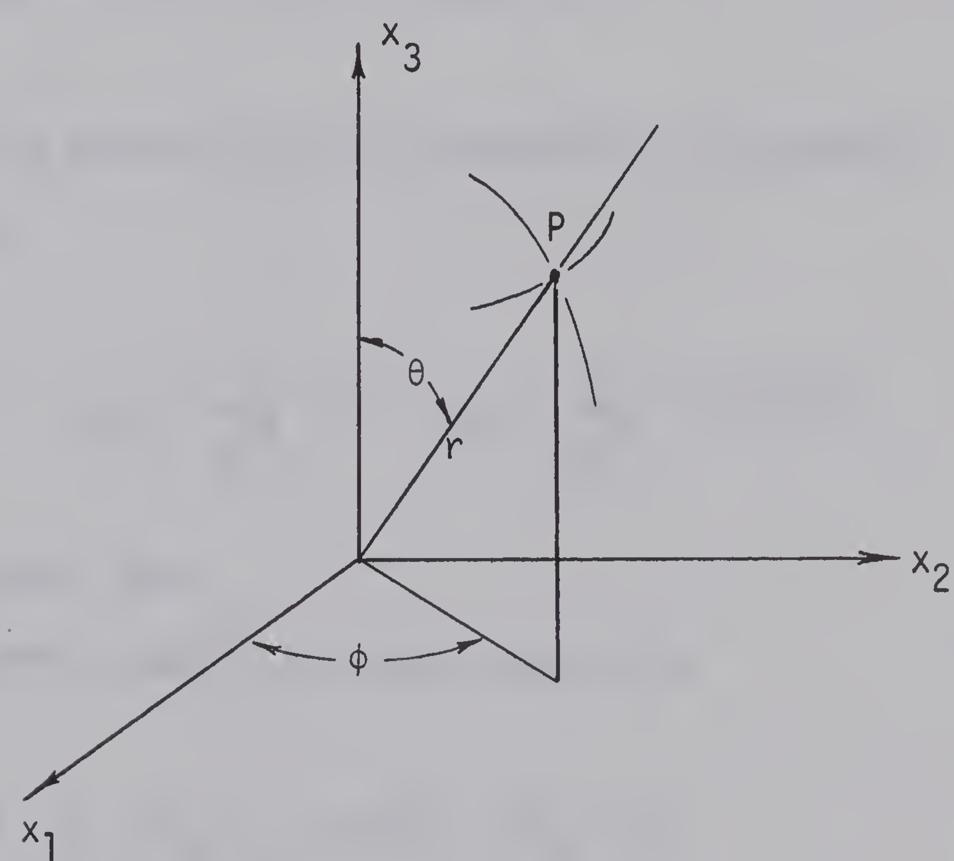


Figure C.1

Spherical Polar Coordinate System

Let the quantities $\xi_i$ be defined by

$$r = \xi_1 ,$$

$$\theta = \xi_2 \ ,$$

and
$$\phi = \xi_3 \ .$$

An element of length ds is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

so that referred to the spherical polar coordinates $\xi_i$ the metric tensor has components

$$g^{11} = g_{11} = 1 \ , \ g_{22} = \frac{1}{g^{22}} = r^2 \ , \ g_{33} = \frac{1}{g^{33}} = r^2 \sin^2\theta$$

with all other components zero.

The Christoffel symbols are thus found to be

$$\Gamma^1_{22} = - r \ , \ \Gamma^1_{33} = - r\sin^2\theta \ , \ \Gamma^2_{12} = \frac{1}{r} \ ,$$

$$\Gamma^2_{33} = - \sin\theta\cos\theta, \ \Gamma^3_{13} = \frac{1}{r} \text{ and } \Gamma^3_{23} = \cot\theta$$

with all others being zero.

Thus the equilibrium equation

$$\tau^{ij}{}_{;i} = 0$$

becomes, referred to the spherical polar coordinates $\xi_i$ ,

$$\tau^{11}{}_{,1} + \tau^{21}{}_{,2} + \tau^{31}{}_{,3} + \cot\theta\tau^{21} + \frac{2}{r}\tau^{11} - r\tau^{22} - r\sin^2\theta\tau^{33} = 0 ,$$

$$\tau^{12}{}_{,1} + \tau^{22}{}_{,2} + \tau^{32}{}_{,3} + \frac{4}{r}\tau^{12} + \cot\theta\tau^{22} - \sin\theta\cos\theta\tau^{33} = 0 ,$$

and $$\tau^{13}{}_{,1} + \tau^{23}{}_{,2} + \tau^{33}{}_{,3} + \frac{4}{r}\tau^{13} + 2\cot\theta\tau^{23} = 0 .$$

# APPENDIX D

## GAUSSIAN QUADRATURE INTEGRATION

The Gaussian quadrature method of numerical integration, which is known for its simplicity and accuracy, is discussed briefly in this appendix.

As a preliminary however, the Lagrangian interpolation polynomial [74] for unequally spaced data must be considered.

If data $y_1, y_2, \ldots, y_n$ are known at points $x_1, x_2, \ldots, x_n$ which are not necessarily equally spaced then a polynomial $\phi_{n-1}(x)$ which is of degree n-1 may be fitted through these points. Such a polynomial is the Lagrangian interpolation polynomial

$$\phi_{n-1}(x) = \frac{(x-x_2)(x-x_3)\ \ldots\ (x-x_n)}{(x_1-x_2)(x_1-x_3)\ \ldots\ (x_1-x_n)}\, y_1$$

$$+ \frac{(x-x_1)(x-x_3)\ \ldots\ (x-x_n)}{(x_2-x_1)(x_2-x_3)\ \ldots\ (x_2-x_n)}\, y_2$$

$$+ \ldots$$

$$+ \frac{(x-x_1)(x-x_2)\ \ldots\ (x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ \ldots\ (x_n-x_{n-1})}\, y_n$$

or

$$\phi_{n-1}(x) = \sum_{i=1}^{n} L_i(x)\, y_i \tag{D.1}$$

where the Lagrangian coefficients $L_i(x)$ may be written as

$$L_i(x) = \frac{\beta(x)}{(x-x_i)\beta'(x_i)}$$

and

$$\beta(x) = (x-x_1)(x-x_2)(x-x_3) \ldots (x-x_n) \; . \qquad (D.2)$$

Since an integral

$$I = \int_a^b f(s)\, ds$$

may always be put in the form

$$I = \int_{-1}^{+1} f(x)\, dx \qquad (D.3)$$

by the change of variable

$$x = \frac{2s - (a+b)}{(b-a)} \; ,$$

only the integral (D.3) is considered in the following.

The question arises, how may the integral I be put in the form

$$\int_{-1}^{+1} f(x) dx = w_1 y(x_1) + w_2 y(x_2) + \ldots + w_n y(x_n) \qquad (D.4)$$

so that it is exact for f(x) as high a degree polynomial as possible.

For n points $x_1$, $x_2$, ... , $x_n$, a polynomial $\phi_{n-1}(x)$ of degree n-1 determined by equation (D.1) may be fitted through the corresponding ordinate points. If the integrand f(x) is a polynomial of degree n-1 or less then $\phi_{n-1}(x)$ and f(x) are identical. However if f(x) is a polynomial of degree 2n-1 or less it may be written as

$$f(x) = \phi_{n-1}(x) + \beta(x)\, z_{n-1}(x) \ , \tag{D.5}$$

where $\beta(x)$ is a polynomial of degree n as given by (D.2) and $z_{n-1}(x)$ is a polynomial of degree n-1. Integration of (D.5) gives

$$\int_{-1}^{+1} f(x)dx = \sum_{i=1}^{n} w_i y(x_i) + R_{n-1} \ ,$$

where

$$w_i = \int_{-1}^{+1} L_i(x)\, dx \tag{D.6}$$

and

$$R_{n-1} = \int_{-1}^{+1} \beta(x) Z_{n-1}(x)\, dx \ .$$

The term $R_{n-1}$ is the error which results from representing the function f(x) by the polynomial $\phi_{n-1}(x)$ . A sufficient condition that $R_{n-1}$ be zero is that $\beta(x)$ be a Legendre polynomial of degree n since the Legendre polynomials, which may be found from

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n ,$$

have the property that

$$\int_{-1}^{+1} P_n(x)\, x^m dx = 0$$

for all $m < n$ .

If $\beta(x)$ is chosen equal to $P_n(x)$ then the values of $x_1$, $x_2$, ..., $x_n$ are known since they are the roots of the polynomial $P_n(x)$. If data are chosen at these n points and the weighting coefficients $w_i$ are found from equation (D.6), the integration formula (D.4) will be exact if f(x) is a polynomial of degree 2n-1 or less. This partly explains the accuracy of the method since most other integration formulae using n data points are exact only if the integrands are polynomials of degree n-1 or less.

The roots $x_i$ of the Legendre polynomials and the weighting factors $w_i$ are readily available in tabulated form [75].

# APPENDIX E

## NEWTON'S INTERPOLATION FORMULAE

Let $y_o$, $y_1$, ..., $y_n$ be the values of the function f(x) at n+1 points $x_o$, $x_1$, ..., $x_n$ which are equally spaced by a distance h. It is possible to approximate the function f(x) by a polynomial $\phi_n(x)$ of degree n so that values of f(x) at intermediate values of x may be estimated. This is the basis of Newton's interpolation formulae.

The derivation of these formulae is simplified by the use of three operators, E the shift operator, $\Delta$ the forward-difference operator, and $\nabla$ the backward-difference operator. These are defined by

$$E\ f(x) = f(x+h)\ ,$$

$$\Delta f(x) = f(x+h) - f(x)\ ,$$

and

$$\nabla f(x) = f(x) - f(x-h)\ .$$

Integer powers of the operators $\Delta$ and $\nabla$ are defined by

$$\Delta^m f(x) = \Delta^{m-1} f(x+h) - \Delta^{m-1} f(x)\ ,$$

and $$\nabla^m f(x) = \nabla^{m-1} f(x) - \nabla^{m-1} f(x-h) ,$$

where m is an integer.

Furthermore integer powers of E are defined by

$$E^m f(x) = f(x+mh)$$

and extending this to non-integer powers gives

$$E^\alpha f(x) = f(x+\alpha h)$$

where $\alpha$ is any real positive number.

Two important relations between the operators E, $\Delta$, and $\nabla$ are

$$\Delta = (E-1) \tag{E.1}$$

and $$\nabla = (1-E^{-1}) .$$

These are easily verified using the definitions of the operators E, $\Delta$, and $\nabla$.

Consider the following expansion based on equation (E.1)

$$E^\alpha = (1+\Delta)^\alpha$$

$$= 1 + \alpha\Delta + \frac{\alpha(\alpha-1)}{2!}\Delta^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Delta^3 + \ldots \qquad (E.2)$$

If $\alpha$ is equal to an integer value n, the above series is finite with n+1 terms. Applying the operator (E.2) at the point $y(x_o)$ gives

$$y(x_o+\alpha h) = [1 + \alpha\Delta + \frac{\alpha(\alpha-1)}{2!}\Delta^2 + \ldots +$$

$$\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\Delta^n]\, y(x_o) + R(x) \qquad (E.3)$$

where

$$R(x) = [\frac{\alpha(\alpha-1)\ldots(\alpha-n)}{(n+1)!} + \frac{\alpha(\alpha-1)\ldots(\alpha-n-1)}{(n+2)!} + \ldots]\, y(x_o) .$$

Denoting by $\phi_n(x)$ the first term on the right hand side of equation (E.3) gives

$$y(x_o+\alpha h) = \phi_n(x) + R(x) .$$

If $\alpha$=n, the term R(x) is zero since the series (E.2) terminates after n terms so that the expression

$$y(x_o+\alpha h) = \phi_n(x)$$

where

$$\phi_n(x) = [1 + \alpha\Delta + \frac{\alpha(\alpha-1)}{2!}\Delta^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}\Delta^n]y(x_o) \quad (E.4)$$

is exact. Equation (E.4) is called Newton's forward interpolation formulae.

A similar expansion of

$$E^{\alpha} = (1-\nabla)^{-\alpha}$$

gives

$$y(x_n+\alpha h) = \phi_n(x) + R(x)$$

where

$$\phi_n(x) = [1 + \alpha\nabla + \frac{\alpha(\alpha+1)}{2!}\nabla^2 + \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!}\nabla^n]y(x_n). \quad (E.5)$$

Equation (E.5) is called Newton's backward interpolation formula.